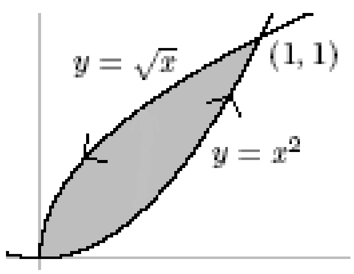
Green's Theorem

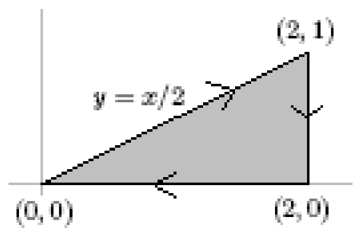
Theorem 1. *Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let D be the region enclosed by C. Suppose that P and Q have continuous partial derivatives on an open region containing D. Then*

$$\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA.$$

Positively oriented means that if you were to walk along the boundary curve, the enclosed region would always be on your *left.* If C is connected, then this corresponds to tracing the curve counterclockwise. Changing the orientation changes the sign. *Piecewise-smooth* means that C may consist of a finite number of curves, each of which is smooth, though they need not fit together smoothly. By *closed* we means the curve starts and ends at the same place, and by *simple* we mean it doesn't intersect itself.

Green's theorem is thought of as a generalization of the fundamental theorem of calculus in the following way. The integral of $P_x - Q_y$ over a 2-dimensional region is given by evaluating P and Q only on the boundary of the region, whereas with usual FTC, an integral of f' over an interval (straight line segment) is given by evaluating f at the boundary of the interval (the two endpoints). The proof of Green's theorem in full generality is a little rough, because one has to make precise the definition of orientation, and deal with possibly some rather strange curves. However the proof in the case where C is anything reasonable is found is most calculus books and is actually quite easy. It involves considering the case where C is a curve given in terms of two functions like in the figures below, and then breaking up the curve into pieces that look like these. Green's theorem is used to compute line integrals over closed curves in the plane. Line integrals over regions such as that obtained by intersecting the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x + z = 1$, where the curve extends into the third dimension, are best handled with Stokes' theorem.





Example: Evaluate $\displaystyle\int_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$, where C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$, oriented positively.

Solution: Notice that the functions $\cos y^2$ and $e^{\sqrt{x}}$ do not have integrals in terms of elementary functions, so this line integral would not be suitable to do evaluate directly. Apply Green's theorem. We calculate $Q_x = 2$ and $P_y = 1$, and get

$$\int_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy = \iint_D (2-1)\,dA = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\,dx = \frac{1}{3}.$$

Example: Evaluate $\displaystyle\int_C xy\,dx + e^{y^2}\,dy$, where C is the triangle with vertices $(0,0)$, $(2,0)$, $(2,1)$ with clockwise orientation.

Solution: Because the orientation is clockwise (i.e., negative) we need a minus sign. We have $Q_x = 0$ and $P_y = x$, and obtain

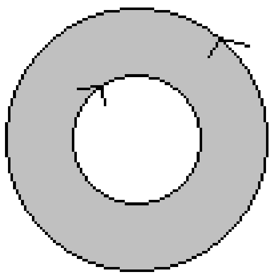
$$\int_C xy\,dx + e^{y^2}\,dy = -\iint_D (0 - x)\,dA = \int_0^2 \int_0^{x/2} x\,dy\,dx = \frac{4}{3}.$$

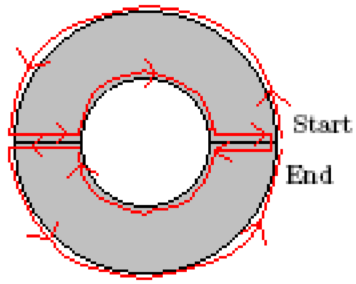
Example: Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = (e^x + x^2 y)\mathbf{i} + (e^y - xy^2)\mathbf{j}$, and C is the circle $x^2 + y^2 = 25$ with counterclockwise orientation.

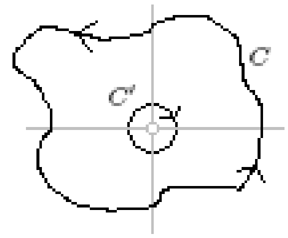
Solution: We've stated the problem a little differently than the first two, but it's really the same thing. It's helpful just to think of it as $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = (e^x + x^2 y)\mathbf{i} + (e^y - xy^2)\mathbf{j}$, and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$. Then using Green's theorem and evaluating the integral in polar coordinates, we get

$$\int_C (e^x + x^2 y)\,dx + (e^y - xy^2)\,dy = \iint_D (-y^2 - x^2) = \int_0^{2\pi} \int_0^5 -r^3\,dr\,d\theta = -\frac{625}{2}\pi.$$

Note that we could also have stated the problem as asking for the work done by moving a particle along C in the force field given by $\mathbf{F}$.







Example: Consider the same $\mathbf{F}$ as in the previous example, but this time let C be the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, oriented positively.

Solution: Green's theorem still works here, even though our curve consists of two disjoint pieces. Note that since the region is oriented positively, the curve is sometimes counterclockwise and sometimes clockwise. This is because for a positive orientation, the region must always be on your left as you traverse the boundary. The figure in the middle gives the idea behind why Green's theorem still holds: we add in lines connecting the circles. Since we traverse both lines once in each direction, their effects cancel out, and it's as if we had never put them there at all. In the end we get

$$\int_C (e^x + x^2 y)\, dx + (e^y - xy^2)\, dy = \iint_D (-y^2 - x^2) = \int_0^{2\pi} \int_1^2 -r^3\, dr\, d\theta = -\frac{15}{2}\pi.$$

Our conclusion here is that Green's theorem works even if the enclosed region has holes (isn't simply-connected).

Example: Show that $\displaystyle\int_C \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy = 2\pi$ for any positively oriented, simple closed path C enclosing the origin.

Solution: Here we get $Q_x - P_y = 0$. It would be really tempting to just say the the line integral evaluates to 0, but the hypotheses of Green's theorem aren't met — P and Q aren't continuous on the enclosed region, hence certainly don't have continuous partials there. However, on any closed region *not* enclosing the origin, Green's theorem implies that the line integral really is 0. We can use this to our advantage as follows. Let C' denote the circle of radius a with clockwise orientation, with a chosen small enough so that the entire circle lies inside of C. Then the region enclosed by the curve $C \cup C'$ does not enclose the origin. Thus Green's theorem gives

$$0 = \int_{C \cup C'} P\, dx + Q\, dy = \int_C P\, dx + Q\, dy + \int_{C'} P\, dx + Q\, dy.$$

That is, the integral over C equals minus the integral over C', so we just have to compute the integral over the circle C', rather than over some arbitrary curve like C. We do this directly by parameterizing the circle by $x = a\sin\theta$, $y = a\cos\theta$, giving $dx = a\cos\theta\, d\theta$ and $dy = -a\sin\theta\, d\theta$ We allow θ to range from 0 to 2π, and we drop the negative sign to account for the fact that the circle is oriented clockwise. We arrive at

$$\int_C \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy = \int_0^{2\pi} \frac{-a\sin\theta}{a^2}(-a\sin\theta)\, d\theta + \frac{a\cos\theta}{a^2}(a\cos\theta)\, d\theta = 2\pi.$$

This is quite closely related to the Cauchy integral formula of complex analysis. Note that the above vector field, $P\mathbf{i} + Q\mathbf{j}$, call it $\mathbf{F}$, is in a sense the only real example of this sort. If $\mathbf{G}$ is another vector field whose partials are equal, but whose line integral is nonzero, then it turns out that $G = \nabla f + k\mathbf{F}$ for some constant k and some function f. Moreover, any line integral over a closed path of the vector field ∇f is necessarily 0 by the fundamental theorem of line integrals.

Example: Compute $\displaystyle\int_C \frac{x-y}{x^2+y^2}\, dx + \frac{x+y}{x^2+y^2}\, dy$ for any positively oriented, simple closed path C enclosing the origin.

Solution: We can write it as

$$\int_C \frac{x}{x^2+y^2}\, dx + \frac{y}{x^2+y^2}\, dy + \int_C \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy.$$

The first integral is zero by the fundamental theorem of line integrals, since the vector field $x/(x^2+y^2)\mathbf{i} + y/(x^2+y^2)\mathbf{j}$ is the gradient of the function $f = \ln(x^2+y^2)/2$, and the second integral evaluates to 2π from the previous example.