How to tell if a series converges or diverges

- **p-Test** — $\sum \frac{1}{n^p}$ *converges if $p > 1$, and diverges if $p \leq 1$.*
- **Geometric Series** — $\sum r^n$ *converges if $|r| < 1$ (if r is between -1 and 1), and diverges otherwise.*
- **nth Term Test** — *Given the series $\sum a_n$, if $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. If the limit is 0, then there is no conclusion.*
  - The idea here is that if the limit is nonzero (let's pretend that it is 2), then after awhile, all the terms of the series are really close to 2, so it's like we're adding 2 to itself an infinite number of times, which causes the series to diverge. However if the limit is zero, the terms are getting smaller and smaller, so the series might converge; but it might not, if the terms don't quite get small enough (for example $\sum \frac{1}{n}$ diverges even though the terms $\frac{1}{n}$ are going to 0).
- **Limit Comparison Test** — *Suppose $\sum a_n$ and $\sum b_n$ are series of positive terms, and $\lim_{n\to\infty} \frac{a_n}{b_n}$ equals a positive number (not 0 or ∞). Then the series either both converge or both diverge.*
  - This is especially useful for a series where the inside is a rational expression (a polynomial divided by another polynomial).
  - To find the right series to compare with, just keep the most important terms of the original series (those terms that are largest for large values of n). For example, from $n^3 + 8n^2 + 1$ we would just keep the n^3, because for large values of n it is much larger than the other terms.
  - It is often enough just to find the right series to compare the given one to, and skip the limit. However, the limit provides a good way to check that we've picked the right series. If we pick the wrong one, the limit would work out to 0 or ∞, indicating our mistake.
- **Ratio Test** — *Let $\sum a_n$ be a series of nonzero terms. Compute $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If the limit is less than 1, then the series converges absolutely. If it is greater than 1, the series diverges. If it equals 1, there is no conclusion.*
  - The ratio test is probably the most useful test. If you're not sure what to use, the ratio test is often a good thing to try.
  - The reason why it works is that having $|\frac{a_{n+1}}{a_n}|$ be eventually less than 1 implies that the terms are getting small pretty quickly, and so the series will converge, whereas if that ratio is larger than 1, the terms are actually getting larger, and the series diverges. If the ratio is exactly equal to 1, then the terms may be getting smaller, but it's not happening all that quickly, and the ratio test doesn't know if it's quite quick enough for the series to converge, so there's no conclusion.
- **Root Test** — *Let $\sum a_n$ be any series. Compute $\lim_{n\to\infty} \sqrt[n]{|a_n|}$. If the limit is less than 1, then the series converges absolutely. If it is greater than 1, the series diverges. If it equals 1, there is no conclusion.*
  - The root test is most useful when most or all of the terms of the series are raised to the nth power or something similar.
  - The conclusion is the same as that of the Ratio Test.

- **Test for Absolute Convergence** — *If $\sum |a_n|$ converges, then $\sum a_n$ converges, too. However, if $\sum |a_n|$ diverges, then $\sum a_n$ might diverge, or it might not.*
  - *Vocabulary* — If the series $\sum |a_n|$ converges, the series is said to be *absolutely convergent*. It may happen that $\sum |a_n|$ diverges, but $\sum a_n$ still converges. In this case, the series is said to be *conditionally convergent*.
  - Some of the convergence tests, like the Integral Test and the comparison tests, can't handle negative terms. The Test for Absolute Convergence tells us that if we look at the series without the negatives, and that series converges, then the original one does, too. If, however, the series without the negatives diverges, then we need to try something else. The Ratio and Root Tests are possibilities, as is the nth Term Test, or, if the series is alternating, the Alternating Series Test may be used.

- **Alternating Series Test** — *Consider the alternating series $\sum (-1)^n a_n$ or $\sum (-1)^{n+1} a_n$, where the a_n are positive terms. These series converge provided*

  1. *The terms a_n are getting smaller (said precisely, $a_{n+1} \leq a_n$ for all n)*
  2. *The terms a_n are tending toward zero (said precisely, $\lim_{n\to\infty} a_n = 0$).*

  If either of these conditions is not met, then there is no conclusion.

  - An alternating series usually contains a $(-1)^n$ or $(-1)^{n+1}$. Written in long form, the terms alternate being added and subtracted, like $1 - 2 + 3 - 4 + 5 - 6 + \ldots$.
  - If the Alternating Series Test fails, then we need to try another test. There's a good chance that if it fails, then the nth Term Test can be used to show that the series diverges.

- **Direct Comparison Test** — *Suppose $\sum a_n$ and $\sum b_n$ are series of positive terms with $a_n \leq b_n$ for all n.*

  1. If $\sum b_n$ converges, then $\sum a_n$ converges, too.
  2. If $\sum a_n$ diverges, then $\sum b_n$ diverges, too.

  - The first statement says (roughly) that if the b_n's add up to a finite number, and the a_n's are all less than the b_n's, then they have to add up to a finite number, too. On the other hand, the second statement says (roughly) that if the a_n's add up to ∞, and the b_n's are all bigger than the a_n's, then the b_n's have to add up to ∞, too.

- **Integral Test** — *If $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, then $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.*
  - Generally, use the Integral Test if you have a series whose terms could easily be integrated, or as a last resort when none of the other tests seem to work.
  - The Integral Test works because the integral gives the area under $f(x)$ to the right of 1, and the sum (roughly) gives an approximation of the area by rectangles of width 1. See your book for pictures illustrating this.

- **Notes**

  1. The starting point of the sum generally doesn't matter too much. Whether we start the sum at $n = 0$, $n = 1$, or $n = 365$ doesn't change whether or not the series will converge or diverge. The first few terms don't determine whether or not the series adds up to ∞, it's the infinite number of later terms that matters.
  2. Constant terms in the series won't affect convergence or divergence, just factor them out of the series. For example, $\sum \frac{3}{4n^2} = \frac{3}{4} \sum \frac{1}{n^2}$.
  3. Several different tests may work for a given series.

Use the lists below to get an idea of which tests to use for which series.

p-Test

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n^9} \qquad \sum_{n=1}^{\infty} n^{-3} \qquad \sum_{n=1}^{\infty} \frac{1}{2n^4} \qquad \sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n}}$$

Geometric Series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \qquad \sum_{n=1}^{\infty} 3^n \qquad \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \qquad \sum_{n=1}^{\infty} \frac{3^{n+1}}{2^{n-1}} \qquad \sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^{n-1}$$

nth Term Test - *In each example below, the limit of the terms is not zero.*

$$\sum_{n=1}^{\infty} \frac{n^2+1}{2n-1} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{2n-1} \qquad \sum_{n=1}^{\infty} \left(2^n+n^2\right) \qquad \sum_{n=1}^{\infty} \frac{e^n}{n^3}, \qquad \sum_{n=1}^{\infty} \sin n$$

Limit Comparison Test - *This is useful when there's a bunch of terms added or subtracted together.*

$$\sum_{n=2}^{\infty} \frac{n}{2n^2-1} \qquad \sum_{n=1}^{\infty} \frac{n^2+3n+4}{n^3+4n^2+1} \qquad \sum_{n=1}^{\infty} \frac{n^5+3}{n^7+4n} \qquad \sum_{n=2}^{\infty} \frac{n}{\sqrt{n^4-1}} \qquad \sum_{n=1}^{\infty} \frac{2^n-1}{3^n-n^2+1}$$

Ratio test - *This is useful when different types of terms are mixed. It works well for all sorts of series.*

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} \qquad \sum_{n=1}^{\infty} \frac{n^2}{3^n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!} \qquad \sum_{n=1}^{\infty} \frac{n^n}{n!} \qquad \sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n \qquad \sum_{n=1}^{\infty} \frac{(-3)^n}{(2n)!}$$

Root Test - *This is most useful when all the terms are raised to the nth power or something similar.*

$$\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n-1}\right)^n \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{(\ln n)^n} \qquad \sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{2n} \qquad \sum_{n=1}^{\infty} \left(2\sqrt[n]{n}+1\right)^n$$

Test for Absolute Convergence - *This is useful for series with negative terms.*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \qquad \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

Alternating Series Test - *Try the Test for Absolute Convergence first. If it doesn't help, then try this.*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \sum_{n=2}^{\infty} \frac{(-1)^{n+1}\ln n}{n+1} \qquad \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2n+4}$$

Direct Comparison Test - *This can be helpful for difficult series, and often in conjunction with other tests.*

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3} \qquad \sum_{n=1}^{\infty} \frac{\arctan n}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2 e^{1/n}}$$

Integral Test - *This is useful if the terms are easily integrated. The other tests are usually easier.*

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \qquad \sum_{n=1}^{\infty} n^2 e^{-n^3} \qquad \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

Examples - Determine the convergence or divergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

   Solution: These are both p-series. The first converges ($p = 3$), and the second diverges ($p = \frac{1}{2}$).

2. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{2^n}$.

   Solution: These are both geometric series. The first converges ($r = \frac{1}{2}$), and the second diverges ($r = \frac{3}{2}$).

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+5n^3+2}$

   Solution: Use the Limit Comparison Test. Using only the most important terms (highest powers), we compare the given series to the series

   $$\sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

   It is a convergent p-series. From this we may conclude that the original series converges. However, if we want to be certain that we've picked the right thing to compare it to, we should check the following limit.

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n^2+n+1}{n^4+5n^3+2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4+n^3+n^2}{n^4+5n^3+2} = 1.$$

   This is a positive number, so our conclusion is valid. If we had gotten 0 or ∞, then our choice would have been bad, and we'd need to make a different choice, or try a different test.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$

   Solution: We'll try the nth Term Test. We have

   $$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0, \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{2n+1} = 0.$$

   The first series diverges because the limit was not 0. However, since the second limit is 0, the nth Term Test has no conclusion for the second series. Instead, use the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ (which is a divergent p-series) to conclude that the second series diverges.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$.

   Solution: Use the Ratio Test. It tells us to compute the following limit.

   $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}} = \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{4}{n+1} = 0.$$

   As this is less than 1, the series converges. In this problem, all the terms are positive, so we may drop the absolute value. When computing the limit, it is often helpful to group the similar terms together, as done above.

6. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+1}\right)^n$

Solution: As the whole series is raised to the nth power, the Root Test seems appropriate. It tells us to compute the following limit. We will drop the absolute value symbols because all the terms are positive.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+3}{5n+1}\right)^n}=\lim_{n\to\infty}\frac{2n+3}{5n+1}=\frac{2}{5}.$$

As this is less than 1, the series converges.

7. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n\,2n}{n^3-1}$ and $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n\,2n}{n^2-1}$

Solution: For the first series, use the Test for Absolute Convergence. If we drop the $(-1)^n$, the resulting series converges by comparison to the p-series $\sum\frac{2}{n^2}$, so the given series must also converge. However, if we drop the $(-1)^n$ from the second series, the resulting series diverges by comparison to the p-series $\sum\frac{2}{n}$. So we can't draw a conclusion.

Instead, we'll try the Alternating Series Test. First check that the terms $\frac{2n}{n^2-1}$ are getting smaller. Notice that the denominator contains an n^2, which grows much faster than the numerator. This indicates that the terms really are decreasing. Now, if we want to be really technical about it, a nice way to show that the terms are decreasing is to show that the function $\frac{2x}{x^2-1}$ is decreasing for $x>2$. We could do this by showing that its first derivative is negative for $x>2$ (which is not hard to do). Secondly, we have $\lim_{n\to\infty}\frac{2n}{n^2-1}=0$, since the higher power is in the denominator. As both conditions of the Alternating Series Test are satisfied, the series must converge.

8. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^2}$

Solution: Use the Direct Comparison Test with the convergent p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. We have $\displaystyle\frac{\cos^2 n}{n^2}\le\frac{1}{n^2}$, since $\cos^2 n\le 1$. So the given series is smaller than a convergent series, and so it must also converge.

9. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

Solution: The Integral Test works here. We compute the integral

$$\int_2^{\infty}\frac{1}{x\ln x}\,dx=\lim_{b\to\infty}\int_2^b\frac{1}{x\ln x}\,dx=\lim_{b\to\infty}\ln(\ln x)\Big|_2^b=\lim_{b\to\infty}\ln(\ln b)-\ln(\ln 2)=\infty.$$

To compute the integral we used the u-sub $u=\ln x$, $du=\frac{1}{x}\,dx$. For the limit, as b gets large, $\ln b$ gets large, too, and hence so does $\ln(\ln b)$ (recall that $\lim_{x\to\infty}\ln x=\infty$).

10. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{e^{1/n}n\ln n}$

Solution: This is different from the previous problem only because of the $e^{1/n}$. This is enough to make it too hard to integrate. However, $e^{1/n}$ is always greater than 1, so that term is just making the denominator larger. If we remove it, we get the convergent series of the previous example. Since putting the $e^{1/n}$ back in just makes the series smaller, by the Direct Comparison Theorem, this series must converge, too.

Problems - Determine the convergence or divergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$

5. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

6. $\displaystyle\sum_{n=1}^{\infty} n^{-1.7}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n^8}{2^n}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln n}$

10. $\displaystyle\sum_{n=1}^{\infty} \arctan n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

12. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

13. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n\ln n}{(n+1)^3}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

18. $\displaystyle\sum_{n=1}^{\infty} \pi^n$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{3\sqrt[3]{n}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$