Mathematics 248 Spring 2007 – Review for Test 3 - Solutions

1. (a) Diverges by the p-Test ($p = 1$).

   (b) Converges by the Alternating Series Test. The terms $\frac{1}{n}$ are getting smaller, and $\lim_{n\to\infty} \frac{1}{n} = 0$, so the two conditions of the test are satisfied.

   (c) Converges by the p-Test ($p = 2$).

   (d) Converges by the Test for Absolute Convergence. The series without the $(-1)^{n+1}$ is a convergent p-series, therefore, the original series must also converge. The Alternating Series Test could also be used.

   (e) Diverges by the nth Term Test. Notice that $2n + 1$ is always odd, so $(-1)^{2n+1} = -1$ for any n. Therefore, $\lim_{n\to\infty}(-1)^{2n+1} = -1 \neq 0$. Since the limit is not 0,the series diverges by the nth term test. In fact, this series is actually adding up -1 forever, so the sum is $-\infty$ (diverges).

   (f) Diverges by the nth Term Test. $\lim_{n\to\infty} \sin n \neq 0$. $\sin n$ oscillates, and therefore has no limit. Since the limit is not 0, the series diverges by the nth term test.

   (g) Converges by the p-Test ($p = 3$). The 6 has no effect on convergence or divergence.

   (h) Diverges by the p-Test ($p = -2$). The nth Term Test can also be used, since $\lim_{n\to\infty} n^2 = \infty \neq 0$. In fact, this series is $1 + 4 + 9 + 16 + \cdots$ which is clearly adding up to ∞.

   (i) Diverges by the Limit Comparison Test to $\sum \frac{n^2}{n^3} = \sum \frac{1}{n}$ (which diverges by the p-Test ($p = 1$)).

   (j) Converges by the Limit Comparison Test to $\sum \frac{\sqrt{n}}{n^5} = \sum \frac{1}{n^{4.5}}$ (which converges by the p-Test ($p = 4.5$)).

   (k) Converges by the Ratio Test. Compute

$$\lim_{n\to\infty} \left| \frac{\dfrac{2^{n+1}}{(n+1)!}}{\dfrac{2^n}{n!}} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1.$$

   (l) Converges by either the Direct Comparison Test or the Limit Comparison Test to the series from the previous problem.

   (m) Converges by the Root Test. Compute

$$\lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{2n-3}{4n+1} \right)^n \right|} = \lim_{n\to\infty} \frac{2n-3}{4n+1} = \frac{2}{4} < 1$$

   (n) Converges by the Ratio Test. Compute

$$\lim_{n\to\infty} \left| \frac{\dfrac{(n+1)^6}{3^{n+1}}}{\dfrac{n^6}{3^n}} \right| = \lim_{n\to\infty} \frac{(n+1)^6}{n^6} \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^6}{3n^6} = \frac{1}{3} < 1.$$

   To compute the limit in the last step, just notice that only the highest power in the numerator will matter, and that will just be n^6 (with a coefficient of 1).

   (o) Converges by the Test for Absolute Convergence. The series without the $(-1)^n$ converges by the Limit Comparison Test to $\sum \frac{n^2}{n^4} = \sum \frac{1}{n^2}$ which is a convergent p-series.

   (p) Converges by the Geometric Series Test. Factor out a 4 from the denominator to recognize that this is a geometric series with $r = \frac{2}{4} < 1$. The Ratio and Root Tests could also be used, but that would be overkill.

   (q) Diverges by the nth Term Test. $\lim_{n\to\infty} \arctan n = \pi/2 \neq 0$, so the series diverges. To compute the limit, the graph of $\arctan x$ might help.

(r) Converges by the p-Test. Rewrite n^{-6} as $\frac{1}{n^6}$. This is a p-series with $p = 6$. The 4 doesn't affect convergence or divergence.

(s) Diverges by the Geometric Series Test. This is a geometric series with $r = \pi \approx 3.14 > 1$.

(t) Converges by the Ratio Test. Compute

$$\lim_{n\to\infty} \frac{(n+1)\left(\frac{5}{6}\right)^{n+4}}{n\left(\frac{5}{6}\right)^{n+3}} = \lim_{n\to\infty} \left(\frac{5}{6}\right) \frac{n+1}{n} = \frac{5}{6} < 1.$$

(u) Diverges by the nth Term Test. $\lim_{n\to\infty} \frac{n}{n+1} = 1$, so because of the $(-1)^n$, the terms will oscillate back and forth close to 1. The limit is, therefore, not 0, and so the series diverges.

(v) Converges. Use the Test for Absolute Convergence. We are thus looking at $\sum \frac{|\cos n|}{n^3}$. Since we know that $|\cos n| \le 1$, we can compare it to the convergent p-series $\sum \frac{1}{n^3}$ using the Direct Comparison Test.

(w) Diverges by the Integral Test. We have

$$\int \frac{1}{x\ln x}\,dx \qquad \begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = x\,du \end{array} \quad \longrightarrow \int \frac{1}{xu}\,x du = \int \frac{1}{u}\,du = \ln|u| = \ln|\ln x|.$$

So we compute

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \ln\ln x\Big|_2^b = \lim_{b\to\infty} \ln\ln b - \ln\ln 2 = \infty.$$

To compute the limit, notice that $\ln\ln b \to \infty$ since $\lim_{x\to\infty} \ln x = \infty$.

(x) Converges by the Ratio Test. Compute

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n+1}{1} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

In the fourth step we canceled the $n+1$ in the numerator with one of the $(n+1)$'s in the denominator. The limit equals $\frac{1}{e} \approx .368 < 1$, so the series converges. One way to do this limit is to use what we learned in section 8.7 — first take the ln of it, then make it into a fraction, use l'Hopital's rule, and finally take e of the answer. A faster way is to look at page 364 of your book.

2. (a) Use the ratio test (or the root test which would be easier).

$$\lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{5^{n+1}}}{\frac{(x-3)^n}{5^n}} \right| = \lim_{n\to\infty} \frac{5^n}{5^{n+1}} \left| \frac{(x-3)^{n+1}}{(x-3)^n} \right| = \frac{|x-3|}{5}.$$

The series converges provided this limit is less than 1, so we have $|x-3|/5 < 1$, which yields $|x-3| < 5$. So we see that the center is 3 and the radius is 5, and therefore the interval of convergence is one of $(-2, 8)$, $[-2, 8)$, $(-2, 8]$, and $[-2, 8]$. We have to check the endpoints to know which it is. The answer is $(-2, 8)$ since the series diverges at both endpoints.

$x = -2$: $\sum \frac{(-2-3)^n}{5^n} = \sum (-1)^n$ diverges by the nth Term Test (or Geometric Series with r=-1).

$x = 8$: $\sum \frac{(8-3)^n}{5^n} = \sum 1$ diverges by the nth Term Test.

(b) Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+2}}{\frac{x^n}{n+1}}\right| = \lim_{n\to\infty}\frac{n+1}{n+2}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}\frac{n+1}{n+2}|x| = |x|.$$

The series converges provided this limit is less than 1, so we have $|x| < 1$. So we see that the center is 0 and the radius is 1. We check the endpoints to get the answer $[-1, 1)$.

$x = -1$: $\sum \frac{(-1)^n}{n}$ converges by the Alternating Series Test.

$x = 1$: $\sum \frac{(1)^n}{n}$ is a divergent p-series ($p = 1$).

(c) Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{(3(n+1))!}}{\frac{(-1)^n x^n}{(3n)!}}\right| = \lim_{n\to\infty}\frac{(3n)!}{(3n+3)!}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}\frac{1}{(3n+3)(3n+2)(3n+1)}|x| = 0.$$

Since 0 is always less than 1, the series will converge for any value of x. The interval is $(-\infty, \infty)$.

(d) Use the ratio test.

$$\lim_{x\to\infty}\frac{(n+1)!x^{2(n+1)}}{n!x^{2n}} = \lim_{n\to\infty} n|x^2| = \infty.$$

Since ∞ is always greater than 1, this series diverges for all values of x, except the center, $x = 0$.

3.

$f(x) = x^{1/2}$	$16^{1/2} = 4$
$f'(x) = \frac{1}{2}x^{-1/2}$	$\frac{1}{2}16^{-1/2} = \frac{1}{8}$
$f''(x) = -\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}16^{-3/2} = -\frac{1}{256}$

Plug these into the Taylor series formula $f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots$ to get

$$\sqrt{x} = 4 + \frac{1}{8}(x-16) + \frac{-1/256}{2!}(x-16)^2 + \cdots = 4 + \frac{1}{8}(x-16) - \frac{1}{512}(x-16)^2 + \cdots.$$

Plug in $x = 18$ to estimate $\sqrt{18}$ as $4 + \frac{2}{8} - \frac{4}{512} \approx 4.2422$. The exact value from a calculator is 4.2426.

4. Use the Taylor series formula $f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots$. We have $c = 3$, and since every derivative of e^x is e^x, we get

$$e^x = e^3 + e^3(x-3) + \frac{e^3}{2!}(x-3)^2 + \frac{e^3}{3!}(x-3)^3 + \cdots = \sum_{n=0}^{\infty}\frac{e^3(x-3)^n}{n!}.$$

Another, shorter way to do the problem is to plug $x - 3$ into the Taylor series for e^x

$$e^{x-3} = \sum_{n=0}^{\infty}\frac{(x-3)^n}{n!}$$

This is a Taylor series for e^{x-3}, but we need a Taylor series for e^x. But notice that $e^{x-3} = e^{-3}e^x$, so we can solve for e^x to get exactly what we found above. (This calculation is valid because the interval of convergence is $(-\infty, \infty)$.)

5. (a) $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots$.

(b) Notice that $\frac{1}{(1+x)^2}$ is the derivative of $\frac{-1}{1+x}$. We found this series in part (a). Therefore,

$$\frac{1}{(1+x)^2} = \frac{d}{dx}\left(\frac{-1}{(1+x)^2}\right) = \frac{d}{dx}(-1+x-x^2+x^3-x^4+\cdots) = 1-2x+3x^2-4x^3+\cdots = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n.$$

(c) Notice that $\frac{1}{(1+x)^3}$ is almost the derivative of $\frac{1}{(1+x)^2}$. In fact, we have

$$\frac{d}{dx}\left(\frac{1}{(1+x)^2}\right) = \frac{-2}{(1+x)^3}.$$

Therefore, if we take the derivative of the series for $\frac{1}{(1+x)^2}$ and multiply by $-\frac{1}{2}$, we will have our answer.

$$-\frac{1}{2}\frac{d}{dx}(1-2x+3x^2-4x^3+5x^4-\cdots) = -\frac{1}{2}(-2+6x-12x^3+20x^4-\cdots) = \sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2}x^n.$$

(d) $\dfrac{1}{1-9x^2} = \sum_{n=0}^{\infty}(-9x^2)^n = \sum_{n=0}^{\infty}(-1)^n 9^n x^{2n} = 1 - 9x^2 + 81x^4 - 729x^6 + \cdots$.

(e) $\dfrac{1}{2-3x} = \dfrac{1}{2(1-\frac{3}{2}x)} = \dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{3}{2}x\right)^n = \sum_{n=0}^{\infty}\dfrac{3^n}{2^{n+1}}x^n = \dfrac{1}{2} + \dfrac{3}{4}x + \dfrac{9}{16}x^2 + \dfrac{27}{32}x^3 + \cdots$.

(f) $\dfrac{x^2}{x-1} = -x^2\dfrac{1}{1-x} = -x^2\sum_{n=0}^{\infty}x^n = -\sum_{n=0}^{\infty}x^{n+2} = -x^2 - x^3 - x^4 - x^5 - \cdots$.

(g) Use the Taylor series for $\sin x$.

$$x^3\sin x^2 = x^3\sum_{n=0}^{\infty}\frac{(-1)^n(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+5}}{(2n+1)!} = x^5 - \frac{x^9}{3!} + \frac{x^{13}}{5!} - \frac{x^{17}}{7!} + \cdots.$$

(h) Use the Taylor series for e^x.

$$e^{-x^2/2} = \sum_{n=0}^{\infty}\frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!} = 1 - \frac{x^2}{2} + \frac{x^4}{4\cdot 2!} - \frac{x^6}{8\cdot 3!} + \cdots.$$

6. (a) First, a power series for $\sin x^2$ is gotten by plugging x^2 into the power series for $\sin x$.

$$\sin x^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots.$$

Now integrate it term by term to get

$$\begin{aligned}\int_0^{.25}\sin x^2\,dx &= \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots\Big|_0^{.25} \\ &= \left(\frac{.25^3}{3} - \frac{.25^7}{42} + \frac{.25^{11}}{1320} - \frac{.25^{15}}{75600} - \cdots\right) - 0\end{aligned}$$

We need at least 4 correct decimal places. Because of rounding, we need the error in our approximation to be less than .00001. Computing the terms, we see that the first equals .0052 and the second equals 1.45×10^{-6}. Since this is term is less than .00001, we can stop with the first term. To four decimal places, the integral is .0052. (Note that the fourth term is 1.23×10^{-14}, so we could get 14 digits of accuracy if we use the first 3 terms.

(b) First we get a power series for $x^2e^{-x^2}\,dx$.

$$x^2e^{-x^2} = x^2\left(1+(-x^2)+\frac{(-x^2)^2}{2!}+\frac{(-x^2)^3}{3!}+\cdots=\right) = x^2-x^4+\frac{x^6}{2!}-\frac{x^8}{3!}+\cdots.$$

Now integrate term by term to get

$$\begin{aligned}\int_0^{.5} x^2e^{-x^2}\,dx &= \frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7\cdot 2!}-\frac{x^9}{9\cdot 3!}+\cdots\Big|_0^{.5}\\ &= \left(\frac{.5^3}{3}-\frac{.5^5}{5}+\frac{.5^7}{14}-\frac{.5^9}{54}+\cdots\right)-0\end{aligned}$$

We need at least 4 correct decimal places. Because of rounding, we need the error in our approximation to be less than .00001. Computing terms, we see that fifth term is the first that is less than .00001. So to be correct to 4 decimal places we just need to add the first 4 terms, and we get .0359.

7. To do these problems we want to see if we can recognize the sum as a familiar power series evaluated at a specific number. The answers are (a) $e^3-1\approx 19.09$, (b) $\arctan 1=\pi/4$, (c) $\sin 1\approx .8414$, (d) $e^{-1}\approx .3679$, (e) $-2/9$, (f) $e^{3/5}\approx 1.822$.

   Part (e) is particularly tricky. None of the series on page 682 have an n in the numerator like this. However, if we look back to the answer to 5(b), we see the series is

$$\frac{1}{(1+x)^2}=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n \qquad x=\frac{1}{2}\longrightarrow\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{2^n}$$

   Compare the series on the right with the given series

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{2^n}$$

   They are awfully similar. The only differences are the $n=0$ vs $n=1$ and the $n+1$ vs n. We can manipulate the series from 5(b) to fix this.

$$\frac{-x}{(1+x)^2}=-x\sum_{n=0}^{\infty}(-1)^n(n+1)x^n=\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^{n+1}.$$

   Writing out the first few terms of this series with $x=\frac{1}{2}$ and comparing it to the series for this problem shows that they are really the same. So the answer is $\frac{-.5}{(1+.5)^2}=-2/9$.

8. To get ln 2, plug 2 into the power series for $\ln x$

$$\ln 2=(2-1)-\frac{(2-1)^2}{2}+\frac{(2-1)^3}{3}-\frac{(2-1)^4}{4}+\cdots=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots.$$

   This won't work for ln 3 because the interval of convergence is $(0,2]$, and 3 is outside of it. It's quite a bit harder to find a nice infinite sum for ln 3.

9. The rectangular equation is $(x-3)^2+(y-4)^2=25$. The parametric equations are $x=3+5\cos t$, $y=4+5\sin t$.

10. (a) is a hyperbola, because of the minus sign. (b) is an ellipse, because of the plus sign, and the coefficients of x^2 and y^2 are different. (c) is a sideways parabola opening to the right, because we can solve for x to get $x=\frac{3}{2}y^2+\frac{1}{2}y$. (d) is a circle. Complete the square to get $4x^2+4(y^2-4y+4)=15+16\rightarrow x^2+(y-2)^2=31/4$. The center is $(0,2)$ and the radius is $\sqrt{31}/2$.

11. The slope of the tangent line is given by $(dy/dt)/(dx/dt)$. We have $dy/dt = 6t$ and $dx/dt = 2$. Therefore, the slope is $3t$. The point $(4, 11)$ corresponds to $t = 2$, which we can get by setting $x = 2t$ equal to the x-coordinate, 4. Plugging in $t = 2$ gives a slope of 6, and therefore the tangent line has equation $y - 11 = 6(x - 4)$ or $y = 6x - 13$.

12. Since the slope is given by $(dy/dt)/(dx/dt)$, we set $dy/dt = 0$ to find where the slope is horizontal ($dy/dt = 2t + 3 = 0$ gives $t = -3/2$), and we set $dx/dt = 0$ to find where the slope is vertical ($dx/dt = -1$ is never 0). So the slope is never vertical, and it is horizontal at $t = -3/2$, which corresponds to the point $(2.5, -2.25)$.

13. For these problems we use the conversions $x = r\cos\theta$, $y = r\sin\theta$, and $x^2 + y^2 = r^2$. For part (a) we get $r\cos\theta = 2$ which becomes $r = 2\sec\theta$. For part (b) we get $r^2 = 3r\sin\theta$, so that $r = 3\sin\theta$, and for part (c) we get $\sqrt{16 - r^2} = r\sin\theta$.

14. (a) Since $r^2 = x^2 + y^2$, we have $r^2 = 4$, and so using $x^2 + y^2 = r^2$ we get $x^2 + y^2 = 4$, (i.e., $r = 2$ is the equation of a circle of radius 4, centered at the origin).

    (b) Multiply both sides by r to get $r^2 = 4r\sin\theta$. Then the equation becomes $x^2 + y^2 = 4y$. This is ok as an answer, but we can do better. Move the $4y$ to the left side, and complete the square to get $x^2 + (y - 2)^2 = 4$, so $y = r\sin\theta$ is the equation of a circle of radius 2, centered at $(0, 2)$.

    (c) Use the trig identity $\sin 2\theta = 2\sin\theta\cos\theta$. So we have $r = 2\sin\theta\cos\theta$. Multiply through by r^2 to get $r^3 = r\sin\theta r\cos\theta$. Therefore the right side becomes $2xy$. We have to be careful with the left side. Since $r^2 = x^2 + y^2$, we have $r = \pm\sqrt{x^2 + y^2}$, so the final answer is $\pm(x^2 + y^2)^{3/2} = 2xy$.

15.