Math 43 Review Notes

[Disclaimer: This is not a complete list of everything you need to know, just some of the topics that gave people difficulty.]

Dot Product

If $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$, then the dot product of $\mathbf{v}$ and $\mathbf{w}$ is given by

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

For example, if $\mathbf{v} = (4, 7, 6)$ and $\mathbf{w} = (2, 3, 9)$, then $\mathbf{v} \cdot \mathbf{w} = 8 + 21 + 54 = 83$. Notice that the answer is always a *number*.

Row Operations

There are three types of row operations:

(1) Adding a multiple of one row to another.
(2) Switching two rows.
(3) Multiplying a row by a constant.

There's a modification of (1) that you can use to avoid fractions. Namely, if you were going to do the operation $row2 - (3/7)row1$, you could instead use $7row2 - 3row1$. This is just the first operation multiplied by the denominator 7.

Any of these operations can be used to solve $A\mathbf{x} = \mathbf{b}$, and in finding inverses, but to find the LU factorization *only* use operations of type 1 (and not the modification).

Echelon and Reduced Row Echelon (rref) forms

A *pivot* is the first nonzero entry in a row which has no other pivots directly above it.

You can identify the echelon form of a matrix by the following properties:

(1) Below each pivot are zeros.
(2) Each pivot is to right of the one above it.
(3) Rows of all zeros (if any at all) must come at the end.

Echelon form is like a lower triangular form for matrices which aren't necessarily square. The reduced row echelon form of a matrix is an echelon form, but now each pivot must be a 1 and there must be zeroes above the pivots as well as below them.

$$\text{Echelon Forms} \quad \begin{pmatrix} \mathbf{x} & x & x \\ 0 & \mathbf{x} & x \\ 0 & 0 & \mathbf{x} \end{pmatrix} \quad \begin{pmatrix} \mathbf{x} & x & x & x \\ 0 & \mathbf{x} & x & x \\ 0 & 0 & 0 & \mathbf{x} \end{pmatrix} \quad \begin{pmatrix} \mathbf{x} & x & x & x \\ 0 & \mathbf{x} & x & x \\ 0 & 0 & \mathbf{x} & x \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \mathbf{x} & x \\ 0 & 0 & x \\ 0 & 0 & \mathbf{x} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{RREF of above} \quad \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \quad \begin{pmatrix} \mathbf{1} & 0 & x & 0 \\ 0 & \mathbf{1} & x & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \quad \begin{pmatrix} \mathbf{1} & 0 & 0 & x \\ 0 & \mathbf{1} & 0 & x \\ 0 & 0 & \mathbf{1} & x \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix}$$

Elimination Matrices

For an operation of the form $rowi - m \cdot rowj$ you get the elimination matrix E_{ij} from the identity matrix by replacing entry (i, j) (that's row i, column j) with $-m$. If you do row operations in the order we've done in class (use the pivot in the first row to get zeroes below it, then use the pivot in the second row to get zeroes below it, etc.), then E_{ij} will have its $-m$ in the same location as the entry that you were trying to zero out. Note that if you did the operation $rowi + m \cdot rowj$, E_{ij} would have a $+m$ instead of $-m$.

The matrix E_{ij} has the effect that E_{ij} times *any* matrix A subtracts m times row j from row i of A.

LU Factorization

If you do a sequence of row operations to reduce a matrix A into lower triangular form U, it can be written as something like $E_{32}E_{31}E_{21}A = U$. Solving this for A gives $A = (E_{21}E_{31}E_{32})^{-1}U$. That big inverse is what we call L, and we find it similarly to finding the elimination matrices. To find L, start with the identity matrix. For each operation of the form $rowi - m \cdot rowj$ replace the (i, j) entry with $+m$ (note signs are the opposite of what they are for elimination matrices since L is an inverse of elimination matrices.) Again, if you do the operations in the order we've done in class, then whatever position you're trying to get a zero in, the corresponding entry in L gets replaced by m.

For example let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & -5 & 14 \end{pmatrix}$$

Reduce A to lower triangular form, indicating the elimination matrices and find the LU Factorization of A.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & -5 & -5 \end{pmatrix} \overset{row2-2row1}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 3 & -5 & -5 \end{pmatrix} \overset{row3-3row1}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & -8 & 11 \end{pmatrix} \overset{row3+8row2}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 27 \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 8 & 1 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -8 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 27 \end{pmatrix}$$

Using LU Factorization To Solve $A\mathbf{x} = \mathbf{b}$

We can write $A\mathbf{x} = \mathbf{b}$ as $LU\mathbf{x} = \mathbf{b}$. Letting $U\mathbf{x} = \mathbf{c}$ we see that we can solve $A\mathbf{x} = \mathbf{b}$ in two steps:

(1) Solve $L\mathbf{c} = \mathbf{b}$.
(2) Solve $U\mathbf{x} = \mathbf{c}$.

Each step is easy, only requiring back substitution and no row operations. For example, use the LU Factorization in the above example to solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (-1, 2, -8)$.

(1) First solve $L\mathbf{c} = \mathbf{b}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -8 & 1 \end{pmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -8 \end{bmatrix}$$

Write this as three equations

$c_1 = -1$
$2c_1 + c_2 = 2$
$3c_1 - 8c_2 + c_3 = -8$

Solve to get $c_1 = -1$, $c_2 = 4$, $c_3 = 27$.

(2) Then solve $U\mathbf{x} = \mathbf{c}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 27 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 27 \end{bmatrix}$$

Write this as three equations

$x_1 + x_2 + x_3 = -1$
$x_2 + 2x_3 = 4$
$27x_3 = 27$

Solve to get $x_1 = -4$, $x_2 = 2$, $x_3 = 1$.

PA=LU Factorization

If you try to to the LU factorization and find that you can't do it without a row exchange, then use the $PA = LU$ factorization. You start with row operations just like in the LU factorization, but as soon as you see you have to flip two rows, flip them instead in the *original matrix* and *start over again* with row operations on the this "new" A to get L and U. Put your row flips into the matrix P. You get P by starting with the identity matrix and flipping the same rows of it that you flipped during your row operations (i.e., if you flipped rows 2 and 3 during row operations, then P is the identity matrix with rows 2 and 3 flipped).

For example let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Find the PA=LU Factorization of A.

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix} \overset{row2-2row1}{\longrightarrow} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

At this point, stop and notice that the (2,2) entry is 0. This should be a pivot. We need this to be nonzero in order to make the (3,2) entry zero, so we have to do a row exchange. Exchange rows 2 and 3 of A and start over.

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{pmatrix} \overset{row2-row1}{\longrightarrow} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 2 & 4 & 1 \end{pmatrix} \overset{row3-2row1}{\longrightarrow} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The last matrix is in lower triangular form. It is our U. We have

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Transposes and Symmetric Matrices

To get A^T, row i of A becomes column i of A^T. Remember the rules for transposes:

(1) $(A^T)^T = A$
(2) $(A \pm B)^T = A^T \pm B^T$
(3) $(AB)^T = B^T A^T$
(4) $(A^{-1})^T = (A^T)^{-1}$

A symmetric matrix is a matrix whose entries are mirror images of each other on either side of the diagonal. Mathematically, they are defined as matrices with $A = A^T$.

$$\begin{pmatrix} a & u & v & w \\ u & b & x & y \\ v & x & c & z \\ w & y & z & d \end{pmatrix}$$

Example: If A and B are symmetric, show ABA is symmetric.

Answer: $(ABA)^T = A^T B^T A^T = ABA$ (using property (3) and the fact that $A = A^T$, $B = B^T$). We have shown $(ABA)^T = ABA$, so it is symmetric.

Example: If A and B are symmetric is $AB(A+B)$ symmetric?

Answer: $[AB(A+B)]^T = (A+B)^T B^T A^T = (A^T + B^T) B^T A^T = (A+B)BA$. Matrix multiplication is not commutative so this is not necessarily the same as $AB(A+B)$. So it is not necessarily symmetric. For example, check that $AB(A+B)$ and $(A+B)BA$ are not the same with

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$$

Vector Spaces and Subspaces

A vector space, roughly speaking, is a set where you can define addition and multiplication by a number in such a way that basic algebraic rules hold (like $c(\mathbf{v}+\mathbf{w}) = c\mathbf{v} + c\mathbf{w}$, and a few others). Examples of vector spaces are all vectors with 2 components, or all vectors with 3 components, etc. Another example is the set of all 2x2 matrices (or 3x3 matrices, etc).

A subspace of a vector space is a set of some of the objects from the vector space which have to satisfy the following two properties:

(1) if $\mathbf{v}$ and $\mathbf{w}$ are in the subspace, then $\mathbf{v}+\mathbf{w}$ has to be in it, too.
(2) if $\mathbf{v}$ is in the subspace and c is a number, then $c\mathbf{v}$ must be in the subspace.

Usually the vectors in the set to be checked all have something in common or look a certain way. To show it really is a subspace you have to check that both properties hold, in other words, check that $\mathbf{v}+\mathbf{w}$ and $c\mathbf{v}$ have that same thing in common. To show something is a subspace you have to use generic vectors, specific examples are not enough. To show something is *not* a subspace however, specific examples are enough, just find a specific example where one of the properties fails.

Example: Show that all vectors (b_1, b_2, b_3) with $b1 + b2 + b3 = 0$ is a subspace of $\mathbb{R}^3$.

Answer: Examples of vectors in the subspace are $(-1, 0, 1)$ or $(3, -1, -2)$. These are vectors whose entries add up to 0. This is what all vectors in the subspace have in common.

First check property 1. Let $\mathbf{c} = (c_1, c_2, c_3)$ and $\mathbf{d} = (d_1, d_2, d_3)$ be any two vectors where $c_1 + c_2 + c_3 = 0$ and $d_1 + d_2 + d_3 = 0$. Then $\mathbf{c} + \mathbf{d} = (c_1 + d_1, c_2 + d_2, c_3 + d_3)$. For this to be in the subspace the entries in $\mathbf{c} + \mathbf{d}$ must add up to 0. This is true since

$$(c_1 + d_1) + (c_2 + d_2) + (c_3 + d_3) = (c_1 + c_2 + c_3) + (d_1 + d_2 + d_3) = 0 + 0 = 0.$$

Net check property 2. Let $\mathbf{c} = (c_1, c_2, c_3)$ be a vector with $c_1 + c_2 + c_3 = 0$ and let n be any number. Then $n\mathbf{c} = (nc_1, nc_2, nc_3)$ is in the subspace since

$$nc_1 + nc_2 + nc_3 = n(c_1 + c_2 + c_3) = n \cdot 0 = 0.$$

Since both properties hold, it is a subspace.

Example: Is the set of all vectors (b_1, b_2, b_3) with all of the entries whole numbers a subspace of $\mathbb{R}^3$?

Answer: No. You can check that the first property works since adding whole numbers gives whole numbers, but property 2 doesn't work. For example, pick the vector $\mathbf{b} = (1, 1, 1)$ and let $n = .5$. Then $n\mathbf{b} = (.5, .5, .5)$ is not in the subspace since its entries are not whole numbers.

Nullspace

The nullspace of a matrix A, denoted $N(A)$ consists of all vectors which are solutions to the equation $A\mathbf{x} = \mathbf{0}$.

Example: Find the nullspace of the following matrix:

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

Answer: First reduce A to rref to get

$$A = \begin{pmatrix} \mathbf{1} & -2 & 0 & -1 & -3 \\ 0 & 0 & \mathbf{1} & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Keep in mind that finding the nullspace is the same as solving $A\mathbf{x} = \mathbf{0}$, so rewrite the above matrix in equation form.

$x_1 - 2x_2 - x_4 - 3x_5 = 0$
$x_3 + 2x_4 - 2x_5 = 0$

Solve these for the pivot variables x_1 and x_3.

$x_1 = 2x_2 + x_4 + 3x_5$
$x_3 = -2x_4 + 2x_5$

Now write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 + 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

The vectors at the last step are what the book calls "special solutions". There is one for each free variable. The null space consists of all linear combinations of these vectors.

Solving $A\mathbf{x} = \mathbf{b}$

Solving $A\mathbf{x} = \mathbf{b}$ consists solving $A\mathbf{x} = \mathbf{0}$ (i.e. finding the nullspace) plus one more step. First start by row reducing the *augmented matrix*. For example let $b = (-5, -5, 12)$ and let A be the matrix in the nullspace example above. Row reducing the augmented matrix to rref, we get:

$$\left(\begin{array}{ccccc|c} \mathbf{1} & -2 & 0 & -1 & -3 & 1 \\ 0 & 0 & \mathbf{1} & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Notice that if the last entry in the last row were not 0, then there would be no solution, as writing this in equation form, the last equation would be 0 = some nonzero number, which is nonsense.

Now to find the solution: First find the nullspace (which we did above) and then find a particular solution. A particular solution is the solution you get by setting the free variables (in this case x_2, x_4, x_5) all equal to 0. What it works out to is just set the pivot variables equal to the entries in the last column in the augmented matrix, and set the free variables equal to zero. Here we get $x_1 = 1, x_3 = 2, x_2 = x_4 = x_5 = 0$. This give the particular solution $(1, 0, 2, 0, 0)$. Adding this to the solution to $A\mathbf{x} = \mathbf{0}$ (the nullspace) translates it into a solution to $A\mathbf{x} = \mathbf{b}$. Thus the complete solution is:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Rank, etc.
A few definitions:

(1) The *rank* of a matrix is the number of pivots.
(2) A matrix has *full column rank* if every column has a pivot.
(3) A matrix has *full row rank* if every row has a pivot.
(4) A *pivot column* is a column which contains a pivot.

To determine each of the above quantities, reduce the matrix to echelon form, from there it is easy.

Column Space

The column space of a matrix A, denoted $C(A)$, consists of all vectors $\mathbf{b}$ which are solutions of $A\mathbf{x} = \mathbf{b}$. Another way to think of it is as all vectors which are linear combinations of the columns of A. If A is simple enough you can determine $C(A)$ just by looking at it, but otherwise reduce A into echelon form to determine the pivot columns. Then $C(A)$ is the set of linear combinations of the pivot columns of A (not the echelon form, but A itself). Another way to determine $C(A)$ is to reduce the augmented matrix with the last column containing entries $b_1, b_2, \ldots$ to an echelon form. Here is an example:

$$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 1 & 3 & 4 & 6 \\ 2 & 6 & 9 & 16 \end{pmatrix}$$

An echelon form of A found by row reduction is

$$\begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From here we see that columns 1 and 3 of A are pivot columns. Thus $C(A)$ is all linear combinations of these columns. In other words $C(A)$ consists of all vectors which can be written in the following form:

$$C(A) = a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix} \quad \text{where } a \text{ and } b \text{ are any numbers.}$$

The other way we can find $C(A)$ is as follows: First, row reduce the following augmented matrix

$$A = \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 1 & 3 & 4 & 6 & b_2 \\ 2 & 6 & 9 & 16 & b_3 \end{array}\right) \overset{row2-row1}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 - b_1 \\ 2 & 6 & 9 & 16 & b_3 \end{array}\right) \overset{row3-2row1}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 - b_1 \\ 0 & 0 & 3 & 12 & b_3 - 2b_1 \end{array}\right)$$

$$\overset{row3-3row2}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_1 - 3(b_2 - b_1) \end{array}\right)$$

For this to have a solution, the bottom right entry in the augmented matrix must be 0. This means that $b_3 = 3b_2 - b_1$ (setting the entry equal to 0 and solving for b_3). Thus any vector in the column space is of the form

$$\begin{bmatrix} b_1 \\ b_2 \\ 3b_2 - b_1 \end{bmatrix}$$

Notice that any vector from the first way of finding $C(A)$ can be written in this form.

Full Row Rank and Full Column Rank

If a matrix has full column rank (i.e. a pivot in every column), then the following are true:
(1) There are no free variables
(2) The null space of the matrix consists only of the vector of all zeros.
(3) $A\mathbf{x} = \mathbf{b}$ has either no solution or exactly one solution.

If a matrix has full row rank (i.e. a pivot in every row), then the following are true:
(1) The number of free variables is equal to the number of columns minus the number of rows.
(2) $A\mathbf{x} = \mathbf{b}$ always has at least one solution.

More About $A\mathbf{x} = \mathbf{b}$

(1) If the rref of a matrix has zero rows, then it's possible that there is no solution. If the rref has no zero rows, then there is certainly a solution. If there are free variables, then either there's no solution or there are infinitely many solutions (and nothing in between). If there are no free variables, then there can't be infinitely many solutions. There's either no solution or one solution in this case.

(2) If A has full row rank and full column rank, then A is a square matrix (same number of rows as columns) and the rref of A is the identity matrix. Thus $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $A^{-1}\mathbf{b}$, and you can find it by row reducing A and back substitution (note that there's no nullspace to worry about). This is the most important case.

(3) If A has full row rank and more columns than rows, then there are no zero rows in the rref, hence there is at least one solution. However, since there are more columns than rows, there are some columns without pivots, hence there are free variables. Thus there are infinitely many solutions.

(4) If A has full column rank with more rows than columns, then there are zero rows in the rref, thus it is possible there is no solution. Since every column has a pivot, there are no free variables, hence there can't be an infinite number of solutions. Thus there is either no solution or one solution.

(5) If A has neither full row rank nor full column rank, then the rref has zero rows and there are free variables. Thus there is either no solution or infinitely many solutions.

(6) Remember, there can never be exactly 2 or exactly 3 solutions. Zero, one, or infinitely many solutions are the only possibilities. (It's a good exercise to try to show why exactly 2 solutions is impossible.)

Linear Independence

A set of vectors is *linearly independent* if none of the vectors can be written as a linear combination of the others. Otherwise we say the vectors are *linearly dependent.* Mathematically this is expressed by saying the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ are linearly independent if $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_n\mathbf{v_n} = \mathbf{0}$ only when $c_1 = c_2 = \ldots = c_n = 0$. This is the same as saying the matrix with these vectors as its columns has full column rank (do you know why?).

If a vector in the set can be written as a linear combination of others, for many applications it is somehow redundant. A linearly independent set has no such redundancies, which is good.

If any of the following happens, then the vectors are linearly *dependent* .

(1) One vector is a multiple of another, or there is an obvious way to combine some of the columns to get another one of the columns.

(2) One of the vectors is the zero vector.

(3) There are more vectors than entries in each vector.

If the vectors pass the above three conditions, then create a matrix with the vectors as its columns. Row reduce the matrix to echelon form. If every column has a pivot, then the vectors are linearly independent. Otherwise they are linearly dependent.

Examples:

$$\text{Let } v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_5 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \quad v_6 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad v_7 = \begin{bmatrix} 12 \\ 5 \\ -2 \end{bmatrix}$$

(1) $\{v_2, v_3\}$ are linearly dependent since v_3 is twice v_2.

(2) $\{v_1, v_3, v_4\}$ are linearly dependent since v_1 is the zero vector.

(3) $\{v_2, v_4\}$ are linearly independent. To see this, make a matrix whose columns are v_2 and v_4 and row reduce it to an echelon form. The echelon form has pivots in every column.

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{1} & 1 \\ 0 & \mathbf{-1} \\ 0 & 0 \end{pmatrix}$$

(4) $\{v_2, v_4, v_5, v_6\}$ are linearly dependent because there are 4 vectors, but only 3 entries in each.

(5) $\{v_2, v_4, v_6\}$ are linearly dependent. To see this, make a matrix whose columns are v_2, v_4, and v_7 and row reduce it to an echelon form. The echelon does not have pivots in every column.

$$\begin{pmatrix} 1 & 1 & 12 \\ 2 & 1 & 5 \\ 3 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{1} & 1 & 12 \\ 0 & \mathbf{-1} & -19 \\ 0 & 0 & 0 \end{pmatrix}$$