Math 43 Review Notes - Chapters 3, 5, & 6

A **basis** for a vector space is a collection of vectors that satisfy:

1. The vectors are linearly independent. (No redundant vectors)
2. Every element in the vector space is a linear combination of the basis vectors.

These say that a basis is a set from which you can obtain every vector in the vector space, and that this set is as small as possible. Many properties of a vector space can be proved just by proving them for the basis vectors. A typical basis has only a few vectors, whereas the vector space probably has infinitely many, so this is a big help.

Example: The vectors $(1, 0)$ and $(0, 1)$ are a basis for $\mathbb{R}^2$, the set of all vectors with two components. It is easy to see that they are linearly independent, and any vector with two components can be written in terms of them. For example,

$$\begin{bmatrix} 7 \\ -5 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and in general, } \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Similarly, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are a basis for $\mathbb{R}^3$. Can you find a similar basis for $\mathbb{R}^n$, the set of all vectors with n components?

A space has many different bases. For example, given a basis, just multiply each basis vector by a constant to get a new basis. In the above example, for instance, $(2, 0)$ and $(0, 2)$ are also a basis for $\mathbb{R}^2$. Note however that all the bases for a space have to have the same number of vectors. This number is called the **dimension** of the space.

Example: Find a basis for the subspace of $\mathbb{R}^3$ consisting of all vectors whose components add up to 0.

Solution: Let $\mathbf{b} = (b_1, b_2, b_3)$ be any vector in the subspace. Then the sum of its components is 0, i.e., $b_1 + b_2 + b_3 = 0$. Solve this for b_1 to get $b_1 = -b_2 - b_3$. Now write

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix} = b_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We have just shown that any vector in the subspace can be written as a linear combination of the vectors $(-1, 1, 0)$ and $(-1, 0, 1)$, and you can easily check that they are linearly independent, thus these two vectors are a basis for the subspace. Since there are two vectors in the basis, the dimension is of the subspace is 2.

Often we will have a set of vectors that we know satisfies property (2) of a basis, i.e. any element in the space is a combination of the vectors. We call such a set a **spanning set**, or we say it **spans** the vector space. Suppose we are given a set of vectors and we want to find a basis for the set they span. They automatically satisfy property (2) of a basis, but not necessarily property (1), since there may be redundant vectors. To get rid of the redundant vectors we can make a matrix with the vectors as its columns and row reduce to find the pivot columns. The pivot columns correspond to the vectors we keep for our basis, while the other columns correspond to the redundant vectors which we will throw away.

Example: Find a basis for the set spanned by the vectors $(1, 2, 3)$, $(1, 4, 4)$, and $(-1, 2, -1)$.

Solution: As we said above, property (2) is automatically satisfied here, so we just have to throw out the redundant vectors. Following the above discussion,

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 4 & 2 \\ 3 & 4 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{1} & 1 & -1 \\ 0 & \mathbf{2} & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

The first two columns are the pivot columns, hence the corresponding vectors $(1, 2, 3)$ and $(1, 4, 4)$ are a basis. Note that depending on which row operations you do, you might get a different answer. But that's ok, as there are many possible bases for a space.

Example: Do the vectors $(1,1,1)$, $(1,2,3)$, and $(1,4,9)$ form a basis for $\mathbb{R}^3$?.

Solution: We have to verify properties (1) and (2) here. We can do this in one big step. Row reduce the matrix with the vectors as its columns. If there is a pivot in every column, then the vectors are linearly independent. If there is a pivot in every row, then the vectors span $\mathbb{R}^3$. (This is true since having a pivot in every row means that $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$, which means that any vector $\mathbf{b}$ in $\mathbb{R}^3$ can be written as a linear combination of the columns of A.)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{1} & 1 & 1 \\ 0 & \mathbf{1} & 3 \\ 0 & 0 & \mathbf{2} \end{pmatrix}$$

There is a pivot in every row and every column, so the vectors do form a basis.

Bases for Nullspace, Row Space and Column Space

Let A be a matrix with echelon form U and rref R.

Nullspace

The nullspace of A consists of all vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0}$. To find it, reduce A to rref R.

$$A = \begin{pmatrix} 1 & 2 & -3 & -9 & 4 \\ 1 & 2 & 0 & 3 & 6 \\ 2 & 4 & -3 & -6 & 12 \end{pmatrix} \longrightarrow R = \begin{pmatrix} \mathbf{1} & 2 & 0 & 3 & 0 \\ 0 & 0 & \mathbf{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

Write $R\mathbf{x} = \mathbf{0}$ in equation form and solve for the pivot variables.

$$\begin{aligned} \mathbf{x_1} + 2x_2 + 3x_4 &= 0 \\ \mathbf{x_3} + 4x_4 &= 0 \\ \mathbf{x_5} &= 0 \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} \mathbf{x_1} &= -2x_2 - 3x_4 \\ \mathbf{x_3} &= -4x_4 \\ \mathbf{x_5} &= 0 \end{aligned}$$

Use this to find an expression for a typical vector in the nullspace.

$$\mathbf{x} = \begin{bmatrix} \mathbf{x_1} \\ x_2 \\ \mathbf{x_3} \\ x_4 \\ \mathbf{x_5} \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ -4x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

So any vector in the nullspace can be written as a combination of the vectors $(-2,1,0,0,0)$ and $(-3,0,-4,1,0)$. It turns out that the vectors one gets at this point are always linearly independent. So these two vectors are a basis. In general, then the vectors one gets at the last step are a basis. Notice that there is one vector for each free variable, so the dimension of the nullspace is equal to the number of free variables (i.e. the number of non-pivot columns). In this example the dimension is 2.

Column Space

The column space of a matrix A consists of all vectors which are linear combinations of its columns. Another way to think of it is as all vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution. The columns of A are a spanning set, so to get a basis we need to throw out any redundant vectors. To do this we row reduce A to an echelon form to see which columns have pivots (any echelon form will do, rref is ok, but not necessary). The pivot columns of A are a basis, and the dimension of the column space is the number of pivots. For example,

$$A = \begin{pmatrix} 1 & 2 & -3 & -9 & 4 \\ 1 & 2 & 0 & 3 & 6 \\ 2 & 4 & -3 & -6 & 12 \end{pmatrix} \longrightarrow U = \begin{pmatrix} \mathbf{1} & 2 & -3 & -9 & 4 \\ 0 & 0 & \mathbf{3} & 12 & 2 \\ 0 & 0 & 0 & 0 & \mathbf{2} \end{pmatrix}$$

Columns 1,3, and 5 are the pivot columns, so they are a basis, i.e $(1,1,2)$, $(-3,0,-3)$, and $(4,6,12)$ are a basis. Make sure to use the columns of A itself. The dimension in this case is 3.

Row Space

The row space of a matrix A consists of all vectors which are linear combinations of its rows. Another way to think of it is as the column space of A^T. The rows of A are a spanning set, so to get a basis we need to throw out any redundant vectors. To do this we row reduce A to an echelon form to see which rows have pivots (as before, any echelon form will do). The pivot rows of A (or U or R) form a basis. The dimension of the row space is the number of pivots. For example,

$$A = \begin{pmatrix} 1 & 2 & -3 & -9 & 4 \\ 1 & 2 & 0 & 3 & 6 \\ 2 & 4 & -3 & -6 & 12 \end{pmatrix} \longrightarrow U = \begin{pmatrix} \mathbf{1} & 2 & -3 & -9 & 4 \\ 0 & 0 & \mathbf{3} & 12 & 2 \\ 0 & 0 & 0 & 0 & \mathbf{2} \end{pmatrix}$$

Rows 1, 2, and 3 are pivot rows, so they are a basis, i.e., $(1, 2, -3, -9, 4)$, $(1, 2, 0, 3, 6)$, and $(2, 4, -3, -6, -12)$ are a basis. Notice here that you could instead use the pivot rows of U or R as a (nicer) basis. The dimension in this case is 3.

Remarks

Row operations don't change the nullspace or the row space; that's why we can use the pivot rows of A, U, or R as a basis for the row space. However, row operations do change the column space, so it's important that we only use the pivot columns of A itself as a basis for the column space of A.

Notice also that the dimensions of the row space and the column space are always the same number r, and r plus the dimension of the nullspace equals the number of columns of A.

Determinants

The determinant of a matrix is a single number that contains a lot of information about the matrix. The determinant of A is denoted either by $\det(A)$ or $|A|$ and it only applies to square matrices (matrices with the same number of rows as columns). We have three different ways of finding determinants. The best way is often to use a combination of the three.

Method 1: Row reduction

(1) The determinant of a triangular matrix (all zeroes above or below the diagonal) is the product of the entries on the diagonal.

(2) Switching two rows switches the sign of the determinant.

(3) Multiplying a row by a number multiplies the determinant by that number, so you'll have to *divide* your answer by that number to cancel it out.

(4) Adding or subtracting a multiple of one row from another doesn't affect the determinant.

(5) Row operations where the row you replace is also multiplied by a number do change the determinant (say you replace row 1 with $3\,\text{row } 1 - 2\,\text{row } 2$, for example). If you replace row j with $m\,\text{row } j$ - $n\,\text{row } k$, then this multiplies the determinant by m, so you have to divide your answer by m to cancel out its effect.

Combining these gives us a method for finding determinants. Row reduce the matrix to a triangular form, and then multiply the entries on the diagonal. Make sure to take into account signs from row switches, and be sure to divide by any multiples you introduced if you do the row operations in (3) or (5).

Example: Find the determinant of $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Solution: Switch rows 1 and 3. The resulting matrix is triangular. The minus sign comes from the row switch.

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -(1 \cdot 1 \cdot 1) = -1$$

Example: Find the determinant of $\begin{pmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 2 & 3 & 8 \end{pmatrix}$.

Solution: Row reduce to a triangular matrix.

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 2 & 3 & 8 \end{vmatrix} \overset{r2-2r1}{=} \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 2 & 3 & 8 \end{vmatrix} \overset{r3-r1}{=} \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & 5 \end{vmatrix} \overset{r3-2r2}{=} \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 2 \cdot 1 \cdot 5 = 10$$

The row operations are of the type in (4) that doesn't affect the determinant.

Example: Find the determinant of $\begin{pmatrix} 3 & 2 & 1 \\ 7 & 5 & 2 \\ 0 & 0 & 4 \end{pmatrix}$.

Solution: Row reduce to a triangular matrix.

$$\begin{vmatrix} 3 & 2 & 1 \\ 7 & 5 & 2 \\ 0 & 0 & 4 \end{vmatrix} \overset{7r1}{=} \frac{1}{7} \begin{vmatrix} 21 & 14 & 7 \\ 7 & 5 & 2 \\ 0 & 0 & 4 \end{vmatrix} \overset{3r2}{=} \frac{1}{3} \cdot \frac{1}{7} \begin{vmatrix} 21 & 14 & 7 \\ 21 & 15 & 6 \\ 0 & 0 & 4 \end{vmatrix} \overset{r2-r1}{=} \frac{1}{3} \cdot \frac{1}{7} \begin{vmatrix} 21 & 14 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{vmatrix} = \frac{1}{3} \cdot \frac{1}{7} \cdot 21 \cdot 1 \cdot 4 = 4$$

Another way to do this using the row operation in (5) would be

$$\begin{vmatrix} 3 & 2 & 1 \\ 7 & 5 & 2 \\ 0 & 0 & 4 \end{vmatrix} \overset{3r2-7r1\to r2}{=} \frac{1}{3} \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{vmatrix} = \frac{1}{3} \cdot 3 \cdot 1 \cdot 4 = 4$$

If you don't mind the fractions, then you could also do this problem just by subtracting (7/3)row 1 from row 2 to reduce to triangular form. This has the benefit of not having any multiples to remember to cancel out.

Method 2: Big Formula

The determinant can be given by a formula just in terms of the entries of the matrix. It is most useful when the matrix is 3×3 or smaller, since for an $n \times n$ matrix the formula has $n!$ terms. (For instance, when $n = 6$, that's 720 terms.)

(1) A 1×1 matrix only has one entry. The determinant is equal to that entry.

(2) For a 2×2 matrix, the formula is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

(3) For a 3×3 matrix the formula is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

This is already a lot to remember, but there is a shortcut *which only works in the* 3×3 *case.*

Example: (*Shortcut for* 3×3) Compute $\begin{vmatrix} 2 & 4 & 1 \\ 3 & 0 & 7 \\ 2 & 5 & 6 \end{vmatrix}$

First copy the indicated entries onto the right side of the matrix. Multiply the entries along each indicated diagonal, and add up the three values you get. This gives the first three terms in the big formula.

$$\begin{vmatrix} 2 & 4 & 1 \\ \mathbf{3} & 0 & 7 \\ \mathbf{2} & \mathbf{5} & 6 \end{vmatrix} \begin{matrix} \\ \mathbf{3} & \\ \mathbf{2} & \mathbf{5} \end{matrix} \quad \longrightarrow (2)(0)(6) + (4)(7)(2) + (1)(3)(5) = 71$$

Next copy the other indicated entries to the left side of the matrix. Multiply the entries along each indicated diagonal, take the negative of each product, and add up the three values you get. This gives the last three terms in the big formula.

$$\begin{matrix} & \\ & \mathbf{7} \\ \mathbf{5} & \mathbf{6} \end{matrix} \begin{vmatrix} 2 & 4 & 1 \\ 3 & 0 & \mathbf{7} \\ 2 & \mathbf{5} & \mathbf{6} \end{vmatrix} \quad \longrightarrow -(1)(0)(2) - (4)(3)(6) - (2)(7)(5) = -142$$

The answer is then 71-142=-71.

(4) There is a formula which works for bigger matrices. If you look at the 3×3 formula, you'll see that for every possible permutation (x, y, z) of $(1, 2, 3)$, there's a term $a_{1x}a_{2y}a_{3z}$ (For example, $(3, 2, 1)$ and $(2, 1, 3)$ are possible permutations; there's 6 total.) Whether we add or subtract a given term is determined by how many steps it takes us to get from $(1, 2, 3)$ to (x, y, z). An even number of steps corresponds to adding, and an odd number corresponds to subtracting. (For example, it takes 3 steps to get from $(1, 2, 3)$ to $(3, 1, 2)$ — $(1, 2, 3) \to (3, 2, 1) \to (3, 1, 2)$.)

To get the 4×4 formula, use permutations of $(1, 2, 3, 4)$ instead of $(1, 2, 3)$. There should be 24 terms, one for each possible permutation. (For instance, $-a_{13}a_{24}a_{32}a_{41}$ is the term corresponding to the permutation $(3, 4, 2, 1)$.) Bigger matrices work similarly. However, with so many terms, the big formula is usually not practical for matrices larger than 3×3.

Method 3: Cofactors

Let M_{ij} be the matrix left over after crossing out row i and column j. The term $C_{ij} = (-1)^{i+j} \det(M_{ij})$ is called a cofactor. We can use cofactors to compute the determinant. The idea is to pick any row or any column of the matrix, and for each entry in that row or column, multiply the entry and its cofactor, then add up all of the products you computed. In terms of formulas,

$$\begin{array}{ll} \det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} & \text{if you expand across row } i, \\ \det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} & \text{if you expand across column } j. \end{array}$$

The $(-1)^{i+j}$ in C_{ij} gives either a plus or minus sign. One way to get the signs right is to remember the sign matrix $\begin{pmatrix} + - + \\ - + - \\ + - + \end{pmatrix}$, or for 4×4, $\begin{pmatrix} + - + - \\ - + - + \\ + - + - \\ - + - + \end{pmatrix}$. For any size matrix, start with a + in the first entry, and the signs alternate from there.

The above formulas may not be very enlightening; the method is best demonstrated with some examples.

Example: Compute the determinant of $\begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ three ways,

(a) by expanding across row 1,
(b) by expanding down column 2,
(c) by expanding across row 3.

Solution:

(a) The formula says

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

We know $a_{11} = 2$, $a_{12} = 3$, $a_{13} = 4$, so we just have to find the cofactors.

$$A = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad \text{Signs} = \begin{pmatrix} + - + \\ - + - \\ + - + \end{pmatrix} \quad M_{11} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad M_{12} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad M_{13} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\det A = 2\begin{vmatrix} 2 & 5 \\ 0 & 6 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 0 & 6 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 2(12 - 0) - 3(6 - 0) + 4(0 - 0) = 6$$

Use the formula (2) from method 2 for evaluating each 2×2 determinant.

Streamlining things a bit, we see each term consists of two parts, first an entry in row 1, added or subtracted according to the corresponding entry in the sign matrix (or by the formula $(-1)^{i+j}$), and second, the determinant of the matrix we get by crossing out the row and column containing the entry.

(b)

$$\begin{pmatrix} 2 & \mathbf{3} & 4 \\ 1 & \mathbf{2} & 5 \\ 0 & \mathbf{0} & 6 \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\det A = -3\begin{vmatrix} 1 & 5 \\ 0 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ 0 & 6 \end{vmatrix} - 0\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} = -3(6-0) + 2(12-0) - 0(10-4) = 6$$

(c)

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ \mathbf{0} & \mathbf{0} & \mathbf{6} \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\det A = 0\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 0(15-8) - 0(10-4) + 6(4-3) = 6$$

We didn't have to write the last two terms in (c) since they were multiplied by 0. As you can see, the easiest of the three to compute is (c). When evaluating a determinant by cofactors, you get to pick the row or column to expand across. Usually it is easiest to expand across the one with the most zeros.

The Fastest Way:

Probably the fastest way to compute determinants, especially 4×4 determinants, is to combine the methods. Use row operations to create a row or a column with a lot of zeroes, and then use cofactor expansion down that row or column.

Example: Find the determinant of $\begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 8 & 7 & 9 \\ 2 & 0 & 1 & 3 \\ 5 & 4 & 3 & 5 \end{pmatrix}$.

Solution: Subtract 2 row 1 from row 2 and subtract row 1 from row 3. These operations don't change the determinant. Then expand down column 2. We get

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ 3 & 8 & 7 & 9 \\ 2 & 0 & 1 & 3 \\ 5 & 4 & 3 & 5 \end{vmatrix} \overset{\substack{r2-2r1 \\ r4-r1}}{=} \begin{vmatrix} 1 & 4 & 2 & 3 \\ 1 & 0 & 3 & 3 \\ 2 & 0 & 1 & 3 \\ 4 & 0 & 1 & 2 \end{vmatrix} \overset{Expand}{=} -4\begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & 3 \\ 4 & 1 & 2 \end{vmatrix}$$

Now compute the 3×3 determinant using any method. The answer is $-4(17) = -68$.

Properties of Determinants

(1) $\det A = 0 \Leftrightarrow A$ has no inverse. In other words: if $\det A = 0$, then you know A has no inverse, and conversely, if you know A has no inverse, then it must be true that $\det A = 0$.

(2) $\det AB = (\det A)(\det B)$.

(3) $\det A^{-1} = 1/\det A$. Try to prove this in one line using property (2) and the fact that $AA^{-1} = I$.

(4) $\det A^T = \det A$. This rule implies that column operations affect the determinant in the same way that row operations do. (Column operations are like row operations, except that you perform them on the columns instead of the rows.)

Cramer's Rule

Cramer's Rule is a method for solving $A\mathbf{x} = \mathbf{b}$ using determinants. Since it uses determinants, it is really only practical for hand computations on small matrices, however it is useful for proving things, as well as for hand computations when row operations would involve a lot of fractions.

Method:

(1) Compute $\det A$. If you get 0, stop. Cramer's rule won't work.

(2) Let B_1 be the matrix you get by replacing column 1 of A with $\mathbf{b}$. Let B_2 be the matrix you get by replacing column 2 of A with $\mathbf{b}$, etc.

(3) Then the solution is given by

$$x_1 = \frac{\det B_1}{\det A}, \qquad x_2 = \frac{\det B_2}{\det A}, \qquad x_3 = \frac{\det B_3}{\det A}, \quad \ldots$$

Example: Use Cramer's rule to solve $A\mathbf{x} = \mathbf{b}$ where $A = \begin{pmatrix} 2 & 1 & 2 \\ 5 & 7 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$.

Solution: Compute the following:

$$\det A = \begin{vmatrix} 2 & 1 & 2 \\ 5 & 7 & 3 \\ 3 & 1 & 2 \end{vmatrix} = -11 \qquad \det B_1 = \begin{vmatrix} \mathbf{1} & 1 & 2 \\ \mathbf{3} & 7 & 3 \\ \mathbf{6} & 1 & 2 \end{vmatrix} = -55$$

$$\det B_2 = \begin{vmatrix} 2 & \mathbf{1} & 2 \\ 5 & \mathbf{3} & 3 \\ 3 & \mathbf{6} & 2 \end{vmatrix} = 17 \qquad \det B_3 = \begin{vmatrix} 2 & 1 & \mathbf{1} \\ 5 & 7 & \mathbf{3} \\ 3 & 1 & \mathbf{6} \end{vmatrix} = 41$$

$$\text{Therefore} \quad x_1 = 5, \qquad x_2 = -\frac{17}{11}, \qquad x_3 = -\frac{41}{11}.$$

Inverses by Cofactors:

We can regard finding A^{-1} as solving the equation $AA^{-1} = I$ for A^{-1}. We can convert this into a system of equations of the form $A\mathbf{x} = \mathbf{b}$ which we can solve by Cramer's Rule. In the end we get the following method for finding A^{-1}. Just as with Cramer's rule, it's usually not practical for hand computations on large matrices, but it is useful for theoretical calculations and hand computations on 2×2 and 3×3 matrices, especially when row operations involve lots of fractions.

Method:

(1) Compute $\det A$. If you get 0, then stop as there is no inverse.

(2) Compute all the cofactors and put them into a matrix C, so that C_{ij} is in row i, column j.

(3) A^{-1} is given by $C^T / \det A$.

One way to do step 2 is to compute all the determinants you get by crossing out a row and a column of A, and put them into a matrix. The determinant calculated by crossing out row i and column j goes into row i, column j of this new matrix. Then get the signs right by using the sign matrix . Don't touch entries which are in the same location as the + signs of the sign matrix . However, do change the sign of entries which are in the same location as the − signs of the sign matrix.

Example: Find the inverse of $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$.

Solution:

(1) Compute $\det A$ to get 5.

(2) Compute all the determinants you get by crossing out a row and a column of A, and put them into a matrix.

$$\begin{matrix} \begin{matrix} 0\,1\,3 \\ 1\,0\,1 \\ 2\,1\,0 \end{matrix} & \begin{matrix} 0\,1\,3 \\ 1\,0\,1 \\ 2\,1\,0 \end{matrix} & \begin{matrix} 0\,1\,3 \\ 1\,0\,1 \\ 2\,1\,0 \end{matrix} \\ \begin{matrix} 0\,1\,3 \\ 1\,0\,1 \\ 2\,1\,0 \end{matrix} & \begin{matrix} 0\,1\,3 \\ 1\,0\,1 \\ 2\,1\,0 \end{matrix} & \begin{matrix} 0\,1\,3 \\ 1\,0\,1 \\ 2\,1\,0 \end{matrix} \\ \begin{matrix} 0\,1\,3 \\ 1\,0\,1 \\ 2\,1\,0 \end{matrix} & \begin{matrix} 0\,1\,3 \\ 1\,0\,1 \\ 2\,1\,0 \end{matrix} & \begin{matrix} 0\,1\,3 \\ 1\,0\,1 \\ 2\,1\,0 \end{matrix} \end{matrix} \longrightarrow \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 & -2 & 1 \\ -3 & -6 & -2 \\ 1 & -3 & -1 \end{pmatrix}$$

Now apply the sign matrix.

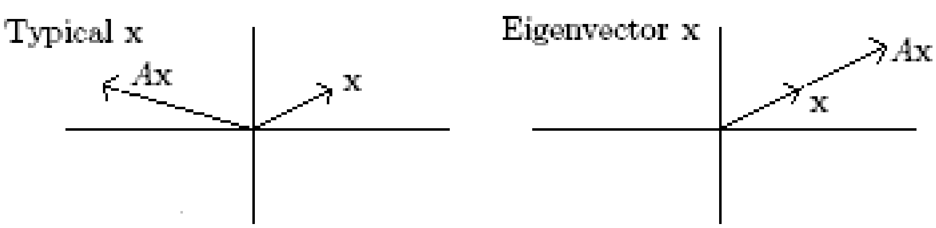
$$\begin{pmatrix} -1 & -2 & 1 \\ -3 & -6 & -2 \\ 1 & -3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} + - + \\ - + - \\ + - + \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{pmatrix} = C$$

(3) Finally, transpose C and divide by $\det A$.

$$A^{-1} = \frac{1}{5} \begin{pmatrix} -1 & 3 & 1 \\ 2 & -6 & 3 \\ 1 & 2 & -1 \end{pmatrix}$$

Eigenvalues and Eigenvectors

Consider multiplying a vector $\mathbf{x}$ by a matrix A. The resulting vector $A\mathbf{x}$ most likely is of a different length and points in a different direction than $\mathbf{x}$.



Those vectors $\mathbf{x}$ for which $A\mathbf{x}$ points in the same direction as $\mathbf{x}$ are called **eigenvectors**. Essentially this means that multiplication by A on $\mathbf{x}$ acts just like multiplying $\mathbf{x}$ by a constant. If we call the constant λ, then we get the equation

$$A\mathbf{x} = \lambda\mathbf{x} \;\text{ if } \mathbf{x} \text{ is an eigenvector.}$$

The constant λ is called an **eigenvalue**.

How to Find Eigenvalues and Eigenvectors

The equation $A\mathbf{x} = \lambda\mathbf{x}$ tells us how to find the eigenvalues and eigenvectors. Subtract $\lambda\mathbf{x}$ from both sides and factor it out to get $(A - \lambda I)\mathbf{x} = \mathbf{0}$. The only way for there to be nonzero eigenvectors (i.e. interesting eigenvectors) is if $A - \lambda I$ is not invertible. Recall that this is true when $\det(A - \lambda I) = 0$. Thus to find the eigenvalues we solve the polynomial equation given by $\det(A - \lambda I) = 0$. For each λ found above there are different eigenvectors. The eigenvectors $\mathbf{x}$ satisfy $(A - \lambda I)\mathbf{x} = \mathbf{0}$, i.e. they are elements of the nullspace of $A - \lambda I$.

Note that $A - \lambda I$ has the same entries as A except that the diagonal entries have λ subtracted from them. For example:

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 2 - \lambda & 4 \\ 3 & 5 - \lambda \end{pmatrix}$$

In summary,

• To find the eigenvalues, compute $\det(A - \lambda I)$. This is a polynomial equation. Set it equal to 0 and solve for λ.

• To find the eigenvectors for an eigenvalue λ, find the nullspace of $A - \lambda I$. The vectors in the nullspace are the eigenvectors corresponding to λ. Do this for all the eigenvalues.

Example: Find the eigenvalues and eigenvectors of $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$.

Solution:

• First find the eigenvalues:

$$\begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$$

So $\lambda = 1, 5$ are the eigenvalues.

• Eigenvectors for $\lambda = 1$: Find the nullspace of $A - 3I$.

$$A - 3I = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \xrightarrow{(1/3)r1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{r2-r1} \begin{pmatrix} \mathbf{1} & 1 \\ 0 & 0 \end{pmatrix} \to x_1 + x_2 = 0 \quad \to x_1 = -x_2$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore the eigenvectors for $\lambda = 1$ are all multiples of $(-1, 1)$.

• Eigenvectors for $\lambda = 5$: Find the nullspace of $A - 5I$.

$$A - 5I = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \xrightarrow{r3+r1} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \xrightarrow{-r1} \begin{pmatrix} \mathbf{1} & -3 \\ 0 & 0 \end{pmatrix} \to x_1 - 3x_2 = 0 \quad \to x_1 = 3x_2$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Therefore the eigenvectors for $\lambda = 5$ are all multiples of $(3, 1)$.

Example: Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$.

Solution:

• First find the eigenvalues: $\det(A - \lambda I)$ might look imposing, but use row operations to simplify it, and then expand across row 1.

$$\begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} \overset{\substack{r1-r3 \\ r2-r3}}{=} \begin{vmatrix} -\lambda & 0 & \lambda \\ 0 & -\lambda & \lambda \\ 2 & 2 & 2-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & \lambda \\ 2 & 2-\lambda \end{vmatrix} + \lambda \begin{vmatrix} 0 & -\lambda \\ 2 & 2 \end{vmatrix}$$

$$= -\lambda(-\lambda(2-\lambda) - 2\lambda) + \lambda(0 - -2\lambda) = 4\lambda^2 - \lambda^3 + 2\lambda^2 = \lambda^2(6 - \lambda)$$

Thus the eigenvalues are $\lambda = 0, 6$.

• Eigenvectors for $\lambda = 0$: Find the nullspace of $A - 0I$.

$$A - 0I = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to x_1 + x_2 + x_3 = 0 \quad \to x_1 = -x_2 - x_3$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the eigenvectors for $\lambda = 0$ are all linear combinations of $(-1, 1, 0)$ and $(-1, 0, 1)$.

• Eigenvectors for $\lambda = 6$: Find the nullspace of $A - 6I$.

$$A - 6I = \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} \mathbf{1} & 0 & -1 \\ 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{array}{c} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{array} \quad \rightarrow \quad \begin{array}{c} x_1 = x_3 \\ x_2 = x_3 \end{array}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore the eigenvectors for $\lambda = 6$ are all multiples of $(1, 1, 1)$.

Remarks about Eigenvalues and Eigenvectors:

- There are a few ways to check your work:
  1. The product of the eigenvalues equals $\det A$.
  2. The sum of the eigenvalues equals the sum of the diagonal entries of A.
  3. Eigenvalues are numbers which make $A - \lambda I$ not invertible. Therefore, if you find that the only eigenvector you get is $\mathbf{0}$ (or equivalently, if you have a pivot in every column), then there is certainly a mistake somewhere.
- The eigenvalues of a triangular matrix are the entries on the diagonal.
- It's possible that the all the eigenvalues are imaginary numbers. This means that the only (real) eigenvector is $\mathbf{0}$.
- Finding the eigenvalues of a $n \times n$ involves solving a polynomial of degree n which is often difficult for $n > 2$.

Diagonalizing a Matrix

A diagonal matrix is a matrix whose entries above and below the diagonal are all zero. Diagonal matrices look like the identity matrix, except that the entries on the diagonal don't have to be all ones. Diagonal matrices are nice to work with because they have so many zero entries. Using eigenvalues and eigenvectors, we can rewrite a matrix in terms of a diagonal matrix. To do this, i.e. to *diagonalize* an $n \times n$ matrix, we need the matrix to have n linearly independent eigenvectors, otherwise we can't do it.

Example:

(1) If all the eigenvalues of a matrix are different, then there will be n linearly independent eigenvectors, so the matrix is diagonalizable.

(2) Suppose a 3×3 matrix has eigenvalues 3 and 4, and in finding the eigenvectors you find for $\lambda = 3$, the nullspace of $A - \lambda I$ is given by $\mathbf{x} = x_2(2, 1, 0) + x_3(-1, 0, 1)$ and for $\lambda = 4$, the nullspace of $A - \lambda I$ is given by $\mathbf{x} = x_3(3, 0, 1)$. The vectors $(2, 1, 0)$, $(-1, 0, 1)$, and $(-3, 0, 1)$ are three linearly independent eigenvectors, so the matrix is diagonalizable.

(3) Suppose in the above example, the nullspace of $A - \lambda I$ for $\lambda = 3$ didn't have the second term. Then we could only find two linearly independent eigenvectors, and so we couldn't diagonalize the matrix.

When you find the nullspace of $A - \lambda I$, the vectors at the last step are a basis for the nullspace. (For example, in (2), the vectors we're referring to are $(2, 1, 0)$ and $(-1, 0, 1)$.) It turns out that the set of all the basis vectors for all the eigenvalues is linearly independent. So you just have to count up the total number of basis vectors you find, and if the total is n, then the matrix is diagonalizable, otherwise it is not.

How to Diagonalize A Matrix

Let Λ be the diagonal matrix whose entries along the diagonal are the eigenvalues of A. Let S be the matrix whose columns are the eigenvectors of A. It is important that the order of the eigenvectors in S corresponds to the order of the eigenvalues in Λ. For example, if the eigenvalue $\lambda = -4$ is in column 3 of Λ, then the eigenvector in column 3 of S must be an eigenvector you found using $\lambda = -4$. We can *diagonalize* A by writing it as

$$A = S\Lambda S^{-1}$$

Example: Diagonalize $A = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$.

Solution: The matrix is upper triangular so the eigenvalues are $\lambda = 2, 5$, the entries on the diagonal.

$$\lambda = 2: \ A - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to x_2 = 0 \to \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = 5: \ A - \lambda I = \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -1/3 \\ 0 & 0 \end{pmatrix} \to x_1 = x_2/3 \to \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2/3 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \qquad S^{-1} = \frac{1}{3} \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}$$

Notice that for the second column instead of using $(1/3, 1)$ we used $(1, 3)$. This is ok since the eigenvectors for $\lambda = 5$ are all the multiples of $(1/3, 1)$. $(1, 3)$ is such a multiple, and we chose it since it has no fractions. This wasn't necessary, but it gives a nicer S. So we diagonalize A as

$$A = S\Lambda S^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}.$$

Using Diagonalization to Find Powers of A

Suppose A is diagonalizable, so that $A = S\Lambda S^{-1}$. Observe the following:

$$A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^2 S^{-1}$$
$$A^3 = A^2 A = (S\Lambda^2 S^{-1})(S\Lambda S^{-1}) = S\Lambda^3 S^{-1}$$

We used the fact that $SS^{-1} = I$ to simplify both expressions. In general we get $A^k = S\Lambda^k S^{-1}$. This is useful because raising diagonal matrices to powers is particularly simple – just raise each diagonal entry to the power. This doesn't usually work with non-diagonal matrices. Thus to compute A^k we only need three multiplications instead of $k + 3$ multiplications.

Example: Use the diagonalization of the matrix in the example above to compute A^k.

Solution:

$$A^k = S\Lambda^k S^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & 5^k \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2^k & 5^k \\ 0 & 3 \cdot 5^k \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^k & (5^k - 2^k)/3 \\ 0 & 5^k \end{pmatrix}$$

We just multiplied all the matrices together. Notice that we brought the 1/3 inside at the last step.