Patterns and Number Theory
Brian Heinold
Mount St. Mary’s University
This is work with Jackie Kearney who researched this for her senior honors project.
What we did

- Plot \( \{(x, y) : f(x, y) \equiv 0 \pmod{n}\} \)
- Various functions \( f(x, y) \) and values of \( n \)
- Usually \( x, y \) between 0 and \( n \) or \( 2n \)
\[(x, y) : xy \equiv 0 \pmod{15}\]
\{(x, y) : xy \equiv 0 \pmod{15}\}
{(x, y) : xy ≡ 0 (mod 15)}
If $xy \equiv 0 \pmod{n}$, then
\[(n - x)y \equiv ny - xy \equiv 0 \pmod{n}.
\]

Similarly $x(n - y) \equiv 0 \pmod{n}$

As $x$ and $y$ are interchangeable, there is symmetry across $y = x$
\{(x, y) : xy \equiv 0 \pmod{n}\} for n = 1 to 30
Get a blank box if \( n \) is prime

\[
x y \equiv 0 \pmod{n} \\
\iff n \mid xy
\]

Euclid’s Lemma \( \Rightarrow n \mid x \) or \( n \mid y \)

But \( 0 < x, y < n \).
Get a grid pattern if \( n = p^2 \) for an odd prime \( p \).

\[
xy \equiv 0 \pmod{p^2}
\]
\[\iff p^2 \mid xy\]
Then \( p \mid xy \).
Euclid’s Lemma implies \( p \mid x \) or \( p \mid y \).
So only get points of form \((ip, jp)\)
\{(x, y) : xy \equiv 0 \pmod{n}\} \text{ for } n = 2 \text{ to } 30
\{(x, y) : x^2 + y^2 \equiv 0 \pmod{n} \text{ for } n = 2 \text{ to } 30\}
Theorem of Fermat: An odd prime is the sum of two squares if and only if it is of the form $4k + 1$. 
$4k + 1$ primes
$4k + 1$ primes (plots in range 0 to $2n$)
\{(x, y) : x^2 + y^2 \equiv 0 \pmod{n} \text{ for } n = 2 \text{ to } 30\}
Other patterns

- $n$ is the sum of two squares iff each $4k + 3$ prime in the prime factorization of $n$ is raised to an even power.
- This explains:
  - Why 21 is also blank
  - Why various types of grids appear
\{(x, y) : (x^2 - 1)(y^2 - 1) \equiv 0 \pmod{n}\}
\{(x, y) : (x^2 - 1)(y^2 - 1) \equiv 0 \pmod{n}\}
\[
\{(x, y) : (x^2 - 1)(y^2 - 1) \equiv 0 \pmod{n}\}
\]
\{(x, y) : xy(x^2 - 4y^2)(4x^2 - y^2) \equiv 0 \pmod{n}\}
\{(x, y) : xy(x^2 - 4y^2)(4x^2 - y^2) \equiv 0 \pmod{n}\}
\{(x, y) : xy(x^2 - 4y^2)(4x^2 - y^2) \equiv 0 \pmod{n}\}
\{(x, y) : (x^2 - y^2)(x^2 - 4y^2)(4x^2 - y^2) \equiv 0 \pmod{n}\}
\[ \{(x, y) : (x^2 - y^2)(x^2 - 4y^2)(4x^2 - y^2) \equiv 0 \pmod{64}\} \]
\{(x, y) : xy(x^4 - y^4) \equiv 0 \pmod{n}\}
\{(x, y) : xy(x^4 - y^4) \equiv 0 \pmod{n}\}\
\begin{itemize}
\item $xy(x^4 - y^4) = xy(x - y)(x + y)(x^2 + y^2)$
\item $30 = 2 \cdot 3 \cdot 5$
\item $x = 3k + r, \ y = 3k' + r', \ r, r' \in \{0, 1, 2\}$.
\begin{itemize}
\item If $r = r'$, then $3 \mid (x - y)$.
\item If $r \neq r'$, then $3 \mid (x + y)$.
\end{itemize}
\item $x = 5k + r, \ y = 5k' + r', \ r, r' \in \{0, 1, 2, 3, 4\}$.
\begin{itemize}
\item If $r = r'$, then $5 \mid (x - y)$.
\item If $(r, r') = (1, 4) \text{ or } (2, 3)$, then $5 \mid (x + y)$
\item If $(r, r') = (1, 3), \ (2, 4)$, then $5 \mid (x^2 + y^2)$.
\end{itemize}
\end{itemize}
$x^2 + y^2 \mod 12$