Questions of Prime Importance
Brian Heinold
A prime number is an integer greater than 1 whose only divisors are 1 and itself.

The first few primes:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, ...
How many primes are there?
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Infinitely many!
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How common are primes? What percent of numbers are primes?

Proportion of integers between 1 and $n$ that are prime $\approx \frac{1}{\ln n}$.

Roughly $\frac{1}{\ln(1000)} = 14.5\%$ between 2 and 1000.

Roughly $\frac{1}{\ln(1000000)} = 7.2\%$ between 2 and 1,000,000.
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Better estimate: Number of primes less than $n$ is approximately

\[ \int_{2}^{n} \frac{1}{\ln x} \, dx \]

It predicts 78628 (versus exact value 78498).
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$2^7 - 1 = 127$
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They are getting large very fast. Are there infinitely many?
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The exponents $n$ in $2^n - 1$ corresponding to Mersenne primes are 2, 3, 5, 7, 13, 17, 19, 31, 61, .... Plotting the logarithm of these numbers gives:
Log of the exponent of Mersenne primes

\[ \log_2(\log_2 M_n) \]
Sophie Germain primes are primes $p$ where $2p + 1$ is also prime.

2 (because $2 \cdot 2 + 1 = 5$ is also prime)
3 (because $2 \cdot 3 + 1 = 7$ is also prime)
5 (because $2 \cdot 5 + 1 = 11$ is also prime)
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But no one knows for sure. These numbers get very big, very fast ($2^{2^{32}} + 1$ has about a billion digits)
Patterns

Red curve: \( n^2 + n + 41 \) (generates primes at a high rate)
Progressions

Are there infinitely many primes that end in 01?

101, 201, 301, 401, 501, 601, 701, 801, 901, 1001, 1101, 1201, \ldots

Dirichlet's Theorem: If \( a \) and \( b \) have no factors in common, then there are infinitely many primes of the form \( ak + b \).

Numbers ending in 01: \( a = 100, b = 1 \)

Numbers ending in 123: \( a = 1000, b = 123 \)

Applies to many other cases as well.
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\textbf{1123, 2123, 3123, 4123, 5123, 6123, 7123, 8123, 9123, 10123, \ldots}
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How about sequences of equally-spaced primes?

3, 5, 7 (each 2 apart)
5, 11, 17, 23, 29 (each 6 apart)

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Longest known is 25 terms long, starting at 43,142,846,595,714,191
Is there always a prime between consecutive squares?

between 1 and 4: 2, 3
between 4 and 9: 5, 7
between 9 and 16: 11, 13
between 16 and 25: 17, 19, 23
A simple question

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between 16 and 25: 17, 19, 23
between 10,000 and 10,201: 23 primes
between 1,000,000 and 1,002,001: 152 primes

No one can prove the fact, though it is very likely true.
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\( \ln(n^2) = 14 \) vs. \( 2n + 1 = 2001 \)
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Gaps between first 100 primes:

1, 2, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2, 4, 14, 4, 6, 2, 10, 2, 6, 6, 4, 6, 6, 2, 10, 2, 4, 2, 12, 12, 4, 2, 4, 6, 2, 10, 6, 6, 6, 2, 6, 4, 2, 10, 14, 4, 2, 4, 14, 6, 10, 2, 4, 6, 8, 6, 6, 4, 6, 8, 4, 8, 10, 2, 10, 2, 6, 4, 6, 8, 4, 2, 4, 12, 8, 4, 8, 4, 6, 12, 2, 18
Twin Primes

*Twin primes* are primes that are 2 apart.

\[\ldots, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, \ldots\]
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There are some partial results:

- Chen Jingrun 1973: There are infinitely many pairs \((p, p + 2)\) where \(p\) is prime and \(p + 2\) is either prime or the product of two primes.
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- **Recent work (Zhang et al. 2013-2014):** There are infinitely many primes \(p\) such that one of \(p + 2, p + 4, \ldots, p + 246\) is also prime.
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- Recent work (Zhang et al. 2013-2014): There are infinitely many primes \( p \) such that one of \( p + 2, p + 4, \ldots, p + 246 \) is also prime.
- Largest known twin prime pair: \( 3756801695685 \cdot 2^{6666669} \pm 1 \) (about 200,000 digits)
Goldbach’s Conjecture: Every even number greater than 2 is the sum of two primes

4 = 2 + 2
6 = 3 + 3
8 = 5 + 3
10 = 5 + 5 or 3 + 7
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$4 = 2 + 2$
$6 = 3 + 3$
$8 = 5 + 3$
$10 = 5 + 5$ or $3 + 7$

$100 = 3 + 97, 11 + 89, 17 + 83, 29 + 71, 41 + 59, 47 + 53$

28 ways to write 1000
127 ways to write 10,000
810 ways to write 100,000
Here is a graph showing how the number of possible ways to write a number as a sum of two primes increases with $n$. (The horizontal axis runs from $n = 4$ to $n = 100,000$ and the vertical axis runs to about 2000.)
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Partial results:

- Chen Jingrun early 1970s: Every sufficiently large even number can be written as a sum $p + q$, where $p$ is prime and $q$ is either prime or a product of two primes.

- Weak Goldbach conjecture Every odd number greater than 7 is the sum of three primes. Seems to have been proved true in 2013.

- The (strong) Goldbach conjecture has been verified by computer for all integers less than $10^{18}$.
The most famous unsolved problem in math is the *Riemann Hypothesis*. 

There is a $1,000,000 prize for solving it. Its solution would answer many questions about primes. There are many "theorems" in math that start out: "If the Riemann Hypothesis is true, then..." If RH is true, it would mean that we have a pretty good understanding of distribution of primes, that they are distributed pretty regularly. If false, then we don't understand primes as well as we thought.
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Details about the Riemann Hypothesis

The zeta function (very important in number theory):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$ 

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \text{ (diverges)}$$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \approx 1.202$$

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$
The Zeta Function has a connection with primes:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}$$

For example:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{10}{11} \times \cdots.$$
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It has many other zeroes whose real parts are all equal to 1/2. The *Riemann Hypothesis* states that there are no other zeroes. The locations of those zeroes have important consequences for what we know about primes.
Thank you for your attention.